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# FUTURES TRADING, RATIONAL EXPECTATIONS, AND THE EFFICIENT MARKETS HYPOTHESIS

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This paper analyzes a model of a futures market in which both pure speculators and producers participate. Traders form rational expectations about the return on holding futures (the spot price) and the amount they will produce, on the basis of diverse private information and the futures price. Constant absolute risk aversion utility functions and normal distributions are assumed in the model. A set of necessary and sufficient conditions is established for the informational efficiency of the futures market, which is taken to mean that the futures price is a sufficient statistic for information about the spot price. In this model the futures price is not in general a sufficient statistic unless there is information available about only one side of the spot market, in which case the sufficient statistic equilibrium is shown to be the only rational expectations equilibrium in which price is a linear function of the information.

## INTRODUCTION

PRICES IN ASSET MARKETS affect the demand for assets in two ways. Firstly and familiarly prices determine the budget constraints which dealers face. Secondly prices reflect dealers' information about the returns on the assets and so aggregate and transmit information between dealers.

These two aspects of price have been recognized and analyzed by a number of authors. The literature on efficient markets and the Martingale hypothesis represents one approach. (See, for example, Samuelson [**18**, **19**] for theory, Fama [**5**] for a survey of empirical work.) This approach uses arbitrage arguments based on a uniform risk premium on the discount rate. More recent work has sought to model the market in more detail, approaching risk aversion through expected utility maximization. It makes explicit use of the rational expectations hypothesis that dealers form correct expectations on the basis of all the information available to them, including the asset prices.

The major issue has been, when is the strong form of the efficient markets hypothesis true; when do prices aggregate all the relevant private and public information perfectly? This has important implications for the incentives to gather information which are discussed later. The welfare implications are not pursued here, but are also important; prices are a guide to resource allocation and so the information conveyed by the price system affects welfare.

A model of a futures market is analyzed in this paper. Informational efficiency is taken to mean that the futures price is a sufficient statistic for the information available about the return on holding futures (the spot price at the date the futures contract must be honored). The model bears many similarities to the futures market models of Grossman and Stiglitz [**10**, Section on 'Prices as Aggregators'] and Danthine [**4**, Section titled 'A parametric example']. The stock

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market model of Grossman [7] is mathematically equivalent to a special case of my model. The similarities include constant absolute risk aversion utility functions and joint normality of all random variables.

The first theorem which I prove establishes a set of necessary and sufficient conditions for the futures price to be a sufficient statistic for information about the spot price. This establishes that the strong symmetry assumptions made about the information structure in the earlier models, and the identical risk aversion assumptions of Danthine [**4**], and Grossman and Stiglitz [**10**] are not necessary for their results. The essential feature of these models is that information is available about only one side of the spot market. I show that, in general, if there is information available about both spot supply and demand the two types of information 'interfere' with each other, and the futures price cannot be a sufficient statistic.

One important feature of the model presented here is that the agents who produce the commodity which is sold on the spot market are also futures traders. Their decisions on futures trades are affected by their beliefs about both the spot price and their own output. If the futures price is a sufficient statistic for information which is gathered about the spot price, there is no incentive for dealers to seek such information if it is costly. However, this observation does not generate the paradox, for the model presented here, which is described by Grossman [7] and Grossman and Stiglitz [**10**]. This paradox poses an existence problem; if dealers find they can learn nothing from their private information which they do not already know from the price, there is no equilibrium in which costly information is collected, because there is no incentive to collect such information. But on the other hand there is no equilibrium in which information is not collected, because then the price is uninformative and so there is an incentive to collect information. However, in the model presented here, whilst it is true that if the futures price is a sufficient statistic for information about the spot price there is no incentive to gather costly information about the spot price, there are incentives for producers to gather information about their own output, and use it in determining their futures trades. The futures price reflects this information, and as aggregate output is a major determinant of the spot price the futures price is also informative about the spot price.

Theorem 1 deals with the existence of an equilibrium in which the price is a sufficient statistic, but does not eliminate the possibility of multiple equilibria. Theorem 2 goes some way towards doing so, showing that if there is information about only one side of the spot market the unique equilibrium in which price is a linear function of the information is the sufficient statistic equilibrium. The proof of Theorem 2 makes extensive use of the 'projection characterization' of the conditional expectation of normal random variables which is set out at the beginning of the proof,[2] thereby avoiding a great deal of matrix algebra.

[2]The underlying mathematics is the projection theorem in Hilbert space, but I conduct the argument entirely in terms of variances and covariances. Futia [**6**] uses Hilbert space theory in a more sophisticated fashion to study rational expectations equilibria in an infinite stationary sequence of speculative markets.

The model is set out in Section 1, the results are stated and proved in Section 2, and discussed in Section 3.

## 1. THE MODEL

### 1.1 *Description*

This is a two period model. A commodity is produced and traded spot at date 1. The spot price is a random variable $\boldsymbol{r}$, which is determined by stochastic influences on spot supply and demand. At a previous date 0 there is a futures market in the commodity. Futures contracts are traded at price $\boldsymbol{p}$ and must be honored when the spot market opens at date 1. The return on holding futures is the spot price $\boldsymbol{r}$.

The $n$ dealers in the market are indexed by $i = 1, 2 \ldots n$. I use the following notation for dealer $i$:

| | | |
|---|---|---|
| $\boldsymbol{a}_i$: | private information | (realized at 0); |
| $k_i$: | risk aversion; | |
| $\boldsymbol{R}_i$; | revenue | (realized at 1); |
| $\boldsymbol{y}_i$: | future sales | (realized at 0); |
| $\boldsymbol{z}_i$: | production | (realized at 1); |

in the two markets:

| | | |
|---|---|---|
| $\boldsymbol{r}$: | spot price | (realized at 1); |
| $\boldsymbol{p}$: | futures price | (realized at 0). |

All these variables apart from risk aversion $k_i$ are random. The dealers make their decisions about how much to sell forward ($\boldsymbol{y}_i$) on the basis of their information, the realization of the random variable $\boldsymbol{a}_i$, and the futures price $\boldsymbol{p}$. Since $\boldsymbol{y}_i$ depends on the random variable $\boldsymbol{a}_i$ it is itself a random variable, as is the market clearing price $\boldsymbol{p}$.

The dealers may be producers or pure speculators. Producers may be uncertain at date 0 about how much they will produce at 1 ($\boldsymbol{z}_i$), but there is nothing they can do to influence $\boldsymbol{z}_i$. All the production decisions have already been made so $\boldsymbol{y}_i$ is the only decision variable. For a pure speculator $\boldsymbol{z}_i = 0$ in all states of the world.

Dealers sell the difference between their output $\boldsymbol{z}_i$ and their futures sales $\boldsymbol{y}_i$ on the spot market at price $\boldsymbol{r}$. The total revenue of dealer $i$ from spot and future trading is[3]

$$\boldsymbol{R}_i = \boldsymbol{y}_i\boldsymbol{p} + (\boldsymbol{z}_i - \boldsymbol{y}_i)\boldsymbol{r}. \tag{1.1}$$

Both future sales $\boldsymbol{y}_i$ and spot sales $\boldsymbol{z}_i - \boldsymbol{y}_i$ may be either positive or negative.

[3] If futures payments are made at 0 and spot payments at 1 the futures price should be thought of as including an implicit discount factor so that $\boldsymbol{R}_i$ is revenue at date 1.

### 1.2 *Assumptions*

I make the following assumptions:

ASSUMPTION 1: *$\boldsymbol{y}_i$ is chosen by dealer i at date* 0 *to maximize the expected utility of revenue conditional upon his private information and the futures price. The utility function displays positive constant absolute risk aversion $k_i$. That is $\boldsymbol{y}_i$ maximizes*[4]

$$E\{-\exp[-k_i(y_i\boldsymbol{p}+(z_i-y_i)\boldsymbol{r})]\,|\,\boldsymbol{a}_i,\boldsymbol{p}_i\}. \tag{1.2}$$

This expression highlights the dual role of the futures price $\boldsymbol{p}$ which enters explicitly into the revenue function and also affects the conditional distribution of $\boldsymbol{r}$ and $z_i$.

The demand side of the spot market is described by the following *ad hoc* assumption.

ASSUMPTION 2. *Spot demand is*

$$\boldsymbol{d}=-\delta\boldsymbol{r}+\boldsymbol{e},\qquad \delta>0, \tag{1.3}$$

*where $\boldsymbol{e}$ is a random variable.*

In addition, we make the following assumption.

ASSUMPTION 3: *The spot market clears with probability* 1.

The mathematical development of the model hinges on the following distributional assumptions:

ASSUMPTION 4: *$(\boldsymbol{a}_1,\dots\boldsymbol{a}_n,z_1,\dots z_n,\boldsymbol{e})$ is a multivariate normal random variable.*

The production of each dealer $z_i$ and the stochastic disturbance to spot demand $\boldsymbol{e}$ must in the nature of things be univariate. $\boldsymbol{a}_i$ may, however, be multivariate, and $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ $(i\neq j)$ may have different dimensions.

ASSUMPTION 5: *Writing $\boldsymbol{a}=(\boldsymbol{a}_1,\dots\boldsymbol{a}_n)$*

$$\operatorname{cov}\left(\boldsymbol{e}-\sum_{i=1}^{n}z_i,\boldsymbol{a}\right)\neq 0. \tag{1.4}$$

*That is, at least one component of the vector is nonzero.*

[4] In fuller notation, given the realization $p$ of $\boldsymbol{p}$ and $a_i$ of $\boldsymbol{a}_i$, $y_i$ is chosen to maximize

$$E\{-\exp[-k_i(y_ip+(z_i-y_i)\boldsymbol{r})]\,|\,\boldsymbol{a}_i=a_i,\boldsymbol{p}=p\}.$$

$\boldsymbol{y}_i$ is the random variable whose realization is $y_i$. The function relating $y_i$ to the realization of $(\boldsymbol{a}_i,\boldsymbol{p})$ depends on the joint distribution of $(\boldsymbol{r},z_i,\boldsymbol{a}_i,\boldsymbol{p})$. The value of the function depends on the realization $(a_i,p)$.

ASSUMPTION 6:

$$\text{(1.5)} \qquad \operatorname{var}\left(\boldsymbol{e} - \sum_{i=1}^{n} z_i \,|\, \boldsymbol{a}\right) > 0.$$

ASSUMPTION 7:

$$\text{(1.6)} \qquad E(z_i \,|\, \boldsymbol{a}_i) = E(z_i \,|\, \boldsymbol{a}) \qquad (i = 1, \ldots n).$$

ASSUMPTION 8:

$$\text{(1.7)} \qquad (4k_i^2/\delta^2)\operatorname{var}\left(\boldsymbol{e} - \sum_{i=1}^{n} z_i\right)\operatorname{var} z_i < 1 \qquad (i = 1, \ldots n).$$

Assumptions 5–8 place restrictions on the distribution of $(\boldsymbol{a}, z_1, \ldots z_n, \boldsymbol{e})$. The role of Assumption 5 is to ensure that there is information available about the spot price—otherwise questions about the ability of the futures price to transmit such information are singularly uninteresting. Assumption 6 ensures that there is insufficient information available to eliminate all uncertainty so that the demand for futures of risk averse dealers remains finite. Assumption 7 says that a producer's expectation of his own output, conditional on his private information, is the same as his expectation conditional on all the information. (Note that this does not imply that he can learn nothing about total production.) Assumption 8 is economically meaningless but ensures that the expected utility integrals converge. (See the Appendix.) It is always satisfied for pure speculators for whom $\operatorname{var} z_i = 0$.

### 1.3 *Rational Expectations Equilibrium*

The concept of rational (or self-fulfilling) expectations equilibrium has become standard in this area. It captures the rational expectations hypothesis (Muth [**16**]) that agents are forming correct expectations given the model and the information available to them. In this case it is an equilibrium in the standard sense that markets clear; but more profoundly it is an equilibrium in that dealers' beliefs about the distribution of the random variables are self-fulfilling so they have no reason to change the way they form expectations.

I define a rational expectations equilibrium price for this model as a random variable $\boldsymbol{p}$ with the two following properties:

I. FUTURES MARKET CLEARING:[5]

$$\text{(1.8)} \qquad \sum_{i=1}^{n} y_i = 0$$

[5] Strictly $\sum_{i=1}^{n} y_i = 0$ with probability 1. Such a proviso should be included everywhere there is an equality between random variables, but it seems unnecessarily pedantic to do so.

where $y_i$ maximizes

$$E\{-\exp[-k_i(y_i p + (z_i - y_i)r)] \mid a_i, p\} \qquad (i = 1, \ldots n).$$

II. PRICE AND INFORMATION: $p$ is a measurable function of the information variables $a$.

The market clearing condition is obvious. The second condition is more subtle; it ensures that the price is unaffected by random variables which do not influence the futures market directly. The price cannot make available information which is otherwise unavailable to everyone in the market.

I shall restrict attention to prices which are linear functions of $a$ (where $a = (a_1, \ldots a_n)$). That is,

$$(1.9) \qquad p = \nu_0 + \sum_{i=1}^{n} \nu_i a_i$$

for some $\nu_0 \in \mathbb{R}$, $\nu_i \in \mathbb{R}^{\dim a_i}$.

As $a$ is normal, all these prices are normal. This makes explicit calculation of $y_i$ possible. These calculations depend on the distribution of $r$ which is determined in the spot market.

### 1.4 *The Spot Market*

Spot sales are the difference between production and futures sales. From (1.3) the spot market clears when

$$\sum_{i=1}^{n} (z_i - y_i) = -\delta r + e.$$

But from (1.8) $\sum_{i=1}^{n} y_i = 0$ so the spot market clearing condition is

$$(1.10) \qquad \sum_{i=1}^{n} z_i = -\delta r + e.$$

Thus as $\delta > 0$

$$(1.11) \qquad r = (1/\delta)\left(e - \sum_{i=1}^{n} z_i\right).$$

This has several useful implications. Most importantly Assumption 4 and (1.11) imply that $(z_1, \ldots z_n, r, a, p)$ is a multivariate normal random variable. It can be shown that given Assumption 5 (1.4) and (1.11)[6]

$$(1.12)\qquad E(\boldsymbol{r}\mid \boldsymbol{a}) \neq E\boldsymbol{r}$$

and from Assumption 6 (1.5) and (1.11)

$$(1.13)\qquad \operatorname{var}(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p}) > 0$$

for any $\boldsymbol{p}$ satisfying (1.9).

### 1.5 *The Futures Market*

I can now calculate explicit functional forms of the futures sales $\boldsymbol{y}_i$ which maximizes expected utility

$$E\{-\exp[-k_i \boldsymbol{R}_i] \mid \boldsymbol{a}_i, \boldsymbol{p}\}$$

where $\boldsymbol{R}_i = \boldsymbol{y}_i\boldsymbol{p} + (\boldsymbol{z}_i - \boldsymbol{y}_i)\boldsymbol{r}$. Dealer $i$ knows the realization of $\boldsymbol{p}$ and $\boldsymbol{a}_i$ at date 0. His expected utility conditional on these realizations depends upon his choice of $\boldsymbol{y}_i$ and the conditional distribution of $(\boldsymbol{z}_i, \boldsymbol{r})$ given $(\boldsymbol{a}_i, \boldsymbol{p})$.

The calculation of the utility maximizing $\boldsymbol{y}_i$ involves some slightly tedious calculations, but no profound mathematics or economics. I have consigned these calculations to the Appendix. The methods of calculation will not be needed in the rest of this paper; the results are:

$$(1.14)\qquad \boldsymbol{y}_i = E(\boldsymbol{z}_i \mid \boldsymbol{a}_i, \boldsymbol{p}) + \alpha_i \boldsymbol{p} - \beta_i E(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p})$$

[6]The simplest way to demonstrate this uses the projections characterization (equations (2.25) and (2.26)) introduced at the beginning of the proof of Theorem 2. Assumption 5 (1.4) and (1.11) imply that

$$\operatorname{cov}(\boldsymbol{r}, \boldsymbol{a},) \neq 0.$$

If $E(\boldsymbol{r}\mid \boldsymbol{a}) = E\boldsymbol{r}$,

$$\operatorname{cov}(\boldsymbol{r}, \boldsymbol{a}) = \operatorname{cov}(\boldsymbol{r} - E\boldsymbol{r}, \boldsymbol{a}) = \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r}\mid \boldsymbol{a}), \boldsymbol{a}) = 0$$

from the projection characterization. Thus $E(\boldsymbol{r}\mid \boldsymbol{a}) \neq E\boldsymbol{r}$. From Assumption 6 (1.5) and (1.11) $\operatorname{var}(\boldsymbol{r}\mid \boldsymbol{a}) > 0$, $\boldsymbol{r} - E(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p})$ is uncorrelated with $(\boldsymbol{a}_i, \boldsymbol{p})$ and therefore, since all the variables are normal, independent of $(\boldsymbol{a}_i, \boldsymbol{p})$. Thus

$$\begin{aligned}\operatorname{var}(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p}) &= E(\boldsymbol{r} - E(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p}))^2 \\ &= E(\boldsymbol{r} - E(\boldsymbol{r}\mid \boldsymbol{a}) - E(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p}) + E(\boldsymbol{r}\mid \boldsymbol{a}))^2 \\ &= E(\boldsymbol{r} - E(\boldsymbol{r}\mid \boldsymbol{a}))^2 + E(E(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r}\mid \boldsymbol{a}))^2 \\ &\geqslant E(\boldsymbol{r} - E(\boldsymbol{r}\mid \boldsymbol{a}))^2 = \operatorname{var}(\boldsymbol{r}\mid \boldsymbol{a}) > 0\end{aligned}$$

as $E(\boldsymbol{r}\mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r}\mid \boldsymbol{a})$ is a linear function of $\boldsymbol{a}$ and so uncorrelated with $\boldsymbol{r} - E(\boldsymbol{r}\mid \boldsymbol{a})$.

where

$$(1.15)\quad \alpha_i = \left[(1 + k_i\rho_i)^2 - k_i^2\sigma_i^2\omega_i^2\right]/k_i\omega_i^2 > 0,$$

$$(1.16)\quad \beta_i = (1 + k_i\rho_i)/k_i\omega_i^2 > 0,$$

and

$$(1.17)\quad \begin{bmatrix} \sigma_i^2 & \rho_i \\ \rho_i & \omega_i^2 \end{bmatrix} = \operatorname{var}(z_i, \boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}).$$

(1.13) ensures that $\omega_i^2 = \operatorname{var}(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) > 0$, so $\alpha_i$ and $\beta_i$ are finite. The positivity of $\alpha_i$ and $\beta_i$ emerges from the calculations. This is economically plausible; *ceteris paribus* futures sales are an increasing function of the futures price and a decreasing function of the return on holding futures. This will be important in the arguments which follow. For a pure speculator $\sigma_i^2 = \rho_i = 0$, $\alpha_i = \beta_i = 1/k_i\omega_i^2$ and $E(z_i \mid \boldsymbol{a}_i, \boldsymbol{p}) = 0$; futures sales are proportional to the difference between the futures price and expected return.

As $(z_i, \boldsymbol{r}, \boldsymbol{a}_i, \boldsymbol{p})$ is a multivariate normal random variable the conditional mean of $(z_i, \boldsymbol{r})$ given $(\boldsymbol{a}_i, \boldsymbol{p})$ depends on both the joint distribution of $(z_i, \boldsymbol{r}, \boldsymbol{a}_i, \boldsymbol{p})$ and the realization of $(\boldsymbol{a}_i, \boldsymbol{p})$, whilst the conditional variance depends only on the joint distribution. (See Anderson [**2**, Chapter 2] for the relevant theorems on the multivariate normal distribution.) Thus $\alpha_i$ and $\beta_i$ depend on the distribution of $(z_i, \boldsymbol{r}, \boldsymbol{a}_i, \boldsymbol{p})$ but not on the realization of $(\boldsymbol{a}_i, \boldsymbol{p})$, so $\alpha_i$ and $\beta_i$ are not random variables. This will be important in proving the theorems.

Summarizing the argument so far, the equilibrium prices in which I am interested are linear functions of the information variables, from (1.9),

$$(1.18)\quad \boldsymbol{p} = \nu_0 + \sum_{i=1}^{n} \nu_i \boldsymbol{a}_i$$

for some $\nu_0 \in \mathbb{R}$ and $\nu_i \in \mathbb{R}^{\dim \mathbf{a}_i}$, $i = 1, \ldots n$. $\boldsymbol{p}$ satisfies the market clearing equation derived from (1.8) and (1.14):

$$(1.19)\quad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}_i, \boldsymbol{p}) + \alpha_i \boldsymbol{p} - \beta_i E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) = 0,$$

where $\alpha_i$ and $\beta_i$ are positive functions of risk aversion and the conditional variance of $(\boldsymbol{a}_i, \boldsymbol{r})$ given $(\boldsymbol{a}_i, \boldsymbol{p})$ given by (1.15)–(1.17).

## 2. SUFFICIENT STATISTIC THEOREMS

### 2.1 *Statements*

These theorems are concerned with the situations in which the futures price aggregates all the information pertinent to the spot price; technically the futures price is a sufficient statistic for the information about the spot price.[7] As all the random variables are normal the conditional distribution of the spot price is completely characterized by the conditional mean and variance. The question is thus when does

$$E(\boldsymbol{r} \mid \boldsymbol{p}) = E(\boldsymbol{r} \mid \boldsymbol{a}) \tag{2.1}$$

and

$$\operatorname{var}(\boldsymbol{r} \mid \boldsymbol{p}) = \operatorname{var}(\boldsymbol{r} \mid \boldsymbol{a}) \tag{2.2}$$

where $\boldsymbol{a} = (\boldsymbol{a}_1, \dots \boldsymbol{a}_n)$?

Theorem 1 gives a necessary and sufficient condition for the existence of a rational expectations equilibrium in which the price is a sufficient statistic. It is easy to show that there is at most one sufficient statistic equilibrium, which raises the question: are there other nonsufficient statistic equilibria? Theorem 2 provides a partial answer.

THEOREM 1: *There is a rational expectations equilibrium in which the price is a sufficient statistic if and only if there are real numbers $\phi$ and $\beta_0$ such that*

$$\sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}) = \phi - \beta_0 E(\boldsymbol{r} \mid \boldsymbol{a}) \tag{2.3}$$

*and*

$$\beta_0 \neq -\sum_{i=1}^{n} \beta_i^* \tag{2.4}$$

*where*

$$\beta_i^* = \frac{(1 + k_i \operatorname{cov}(z_i, \boldsymbol{r} \mid \boldsymbol{a}))}{k_i \operatorname{var}(\boldsymbol{r} \mid \boldsymbol{a})}\,. \tag{2.5}$$

*No more than one sufficient statistic rational expectations equilibrium exists. If such an equilibrium does exist the equilibrium price is a linear function of* $\boldsymbol{a}$.

[7] A sufficient statistic can be defined in a variety of ways. The standard factorization criterion is that $\boldsymbol{p}$ is a sufficient statistic for the information $\boldsymbol{a}$ about $\boldsymbol{r}$ if the conditional density of $\boldsymbol{a}$ given $\boldsymbol{r}$, $g(\boldsymbol{a} \mid \boldsymbol{r})$, can be factored as

$$g(\boldsymbol{a} \mid \boldsymbol{r}) = g_1(\boldsymbol{p}, \boldsymbol{r})\, g_2(\boldsymbol{a})$$

(see Mood and Graybill [**15**, p. 171]). Grossman [**9**, Appendix] shows that this condition is equivalent to

$$E[F(\boldsymbol{r}) \mid \boldsymbol{a}] = E[F(\boldsymbol{r}) \mid \boldsymbol{p}]$$

for all measurable real $F$. In particular this is true where $F$ is a step function. Thus the conditional distribution function of $\boldsymbol{r}$ given $\boldsymbol{a}$ and given $\boldsymbol{p}$ are the same. When $(\boldsymbol{r}, \boldsymbol{a})$ is normal and $\boldsymbol{p}$ is a linear function of $\boldsymbol{a}$, both conditional distributions are normal and entirely specified by the conditional mean and variance (Anderson [**2**, p. 29]), so (2.1) and (2.2) imply that $\boldsymbol{p}$ is a sufficient statistic.

THEOREM 2: *If* (2.3) *holds with*

$$(2.6) \qquad \beta_0 \geqslant 0,$$

*the only rational expectations equilibrium which exists with price a linear function of* $\boldsymbol{a}$ *is the sufficient statistic equilibrium.*

The conditions of the theorems need an economic interpretation. It is helpful to distinguish three different cases:

(i) (2.3) holds with $\beta_0 = \delta$. Then rewriting (2.3),

$$(2.7) \qquad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}) = \phi - \delta E(\boldsymbol{r} \mid \boldsymbol{a}).$$

From the spot market cleaning condition (1.10),

$$(2.8) \qquad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}) = E(\boldsymbol{e} \mid \boldsymbol{a}) - \delta E(\boldsymbol{r} \mid \boldsymbol{a}).$$

(2.7) and (2.8) taken together imply that

$$(2.9) \qquad E(\boldsymbol{e} \mid \boldsymbol{a}) = E\boldsymbol{e} = \phi.$$

The information $\boldsymbol{a}$ is independent of the stochastic term in spot demand $\boldsymbol{e}$ and refers only to spot supply $\sum_{i=1}^{n} z_i$.

In this case the model is almost a generalization of that of Grossman and Stiglitz [**10**, section on 'Prices as Aggregators']. The models differ in that I allow for the presence of pure speculators as well as producers, and do not insist on symmetrical distributions and risk aversion. Grossman and Stiglitz assume that given all the information there is no uncertainty about the spot price whilst I assume that there is always residual uncertainty. This leads to the breakdown of the market in the Grossman-Stiglitz example, as the absence of uncertainty removes any incentive to futures trading.

The model is also similar to that of Danthine [**4**] (a parametric example); again it differs in the greater generality of its risk aversion and distributional assumptions. Danthine analyzes the effect of futures trading on production decisions; this issue is not considered here.

(ii) (2.3) holds with $\beta_0 = 0$. Then

$$(2.10) \qquad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}) = \sum_{i=1}^{n} Ez_i = \phi.$$

The information $\boldsymbol{a}$ is independent of total output $\sum_{i=1}^{n} z_i$ and refers only to spot demand $\boldsymbol{e}$.

In this case the model is mathematically equivalent to generalization of Grossman's [**7**] model of a stock market, differing from Grossman's model in the greater generality of the information structure.

(iii) (2.7) holds with $0 \neq \beta_0 \neq \delta$. In this case from the spot market clearing condition (1.10) and (2.3),

$$(2.11)\quad \left(\sum_{i=1}^{n} E(z_i\,|\,\boldsymbol{a})\right)(1-\beta_0/\delta) = \phi - (\beta_0/\delta)E(\boldsymbol{e}\,|\,\alpha).$$

This implies that the conditional expectations of spot supply and demand are perfectly correlated. It is difficult to imagine why this should be so; the economically interesting cases (i) and (ii) in which information is available about only one side of the spot market.

The role of the additional conditions in Theorem 1 ((2.4) and (2.5)) will become apparent in the course of the proof. Note that if these conditions fail to hold a small change in one the parameters of the model (e.g., risk aversion) would make them hold. Thus Theorem 1 is generically true if (2.3) holds.

The condition for Theorem 2 (2.6) holds when information is available about only one side of the spot market. (Recall from Assumption 2 (1.3) that $\delta > 0$.) If (2.3) holds with information available about both sides of the spot market, (2.6) implies that the spot price is negatively correlated with supply. This seems reasonable, although it may not be true if $E(\boldsymbol{e}\,|\,\boldsymbol{a})$ is positively correlated with $\sum_{i=1}^{n} E(z_i\,|\,\boldsymbol{a})$.

## 2.2 *Proof of Theorem* 1

The theorem is proved by using Grossman's **[9]** concept of an artificial economy. In this economy dealers pool all their information prior to trading; and trade on the basis of the conditional expectation of $(z_i, \boldsymbol{r})$ given all the information $(\boldsymbol{a})$, rather than $(\boldsymbol{a}_i, \boldsymbol{p})$. $\boldsymbol{p}^*$ is the Walrasian equilibrium price for this economy. The proof proceeds by showing that (2.3) is a necessary and sufficient condition for the artificial economy price to be a sufficient statistic, and then arguing in a similar fashion to Grossman that there is a rational expectations equilibrium in which the price is a sufficient statistic if and only if the artificial economy price is a sufficient statistic. The equilibrium price is the artificial economy price.

If the artificial economy price $\boldsymbol{p}^* = p^*$, and $\boldsymbol{a} = a$, the futures sale $y_i^*$ is chosen to maximize

$$(2.12)\quad E\{-\exp[-k_i(y_i^* p^* + (z_i - y_i^*)\boldsymbol{r})]\,|\,\boldsymbol{a} = a\}.$$

From (1.19)

$$(2.13)\quad y_i = E(z_i\,|\,\boldsymbol{a} = a) + \alpha_i^* p^* - \beta_i^* E(\boldsymbol{r}\,|\,\boldsymbol{a} = a).$$

$\boldsymbol{p}^*$, the Walrasian equilibrium price for the artificial economy, satisfies

$$(2.14)\quad \sum_{i=1}^{n} E(z_i\,|\,\boldsymbol{a}) + \alpha_i^* \boldsymbol{p}^* - \beta_i^* E(\boldsymbol{r}\,|\,\boldsymbol{a}) = 0.$$

$\alpha_i^*$ and $\beta_i^*$ are the functions of $\text{var}(z_i, \boldsymbol{r} \mid \boldsymbol{a})$, set out in (1.15) and (1.16). Note that $\alpha_i^*$ and $\beta_i^*$ are positive real numbers, not random variables. Also $\beta_i^*$ in (2.5) and (2.13) are the same.

(2.14) implies that $\boldsymbol{p}^*$ is a linear function of $\boldsymbol{a}$. A necessary and sufficient condition for $\boldsymbol{p}^*$ to be a sufficient statistic for the information conveyed by $\boldsymbol{a}$ about $\boldsymbol{r}$ is[8]

(2.15) $\quad E(\boldsymbol{r} \mid \boldsymbol{p}^*) = E(\boldsymbol{r} \mid \boldsymbol{a}).$

$E(\boldsymbol{r} \mid \boldsymbol{p}^*)$ is a linear function of $\boldsymbol{p}^*$, so (2.15) implies that $E(\boldsymbol{r} \mid \boldsymbol{a}) = \theta_0 + \theta_1 \boldsymbol{p}^*$ for some $(\theta_0, \theta_1)$. If $\theta_1 = 0$, $E(\boldsymbol{r} \mid \boldsymbol{a}) = \theta_0 = E\boldsymbol{r}$, contradicting (1.12). Thus (2.15) implies that

(2.16) $\quad \boldsymbol{p}^* = \gamma_0 + \gamma_1 E(\boldsymbol{r} \mid \boldsymbol{a})$

for some real $(\gamma_0, \gamma_1), \gamma_1 \neq 0$. It is straightforward to show that (2.16) implies (2.15) by using the standard formula. Thus (2.15) and (2.16) are equivalent.

(2.14) defines $\boldsymbol{p}^*$. Given (2.14), (2.16) can hold if and only if

(2.17) $\quad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}) = \phi - \beta_0 E(\boldsymbol{r} \mid \boldsymbol{a})$

where

(2.18) $\quad \phi = -\gamma_0 \sum_{i=1}^{n} \alpha_i^*$

and

(2.19) $\quad \beta_0 = \gamma_1 \sum_{i=1}^{n} \alpha_i^* - \sum_{i=1}^{n} \beta_i^*.$

Also

(2.20) $\quad \gamma_1 \neq 0$ if and only if $\beta_0 \neq -\sum_{i=1}^{n} \beta_i^*.$

[8] If $(x_1, x_2)$ is a multivariate normal random variable, $x_1 - E(x_1 \mid x_2)$ is independent of $x_2$. (This can be readily checked from the conditional distribution formula, recalling that uncorrelated normal random variables are independent.) Thus

$$\text{var}(x_1 \mid x_2) = E\left[(x_1 - E(x_1 \mid x_2))^2 \mid x_2\right] = E(x_1 - E(x_1 \mid x_2))^2.$$

Therefore if $(x_1, x_2')$ is also multivariate normal and

$$E(x_1 \mid x_2') = E(x_1 \mid x_2),$$

then

$$\text{var}(x_1 \mid x_2) = \text{var}(x_1 \mid x_2').$$

Thus (2.1) implies (2.2), and so (2.1) (repeated as (2.15)) is necessary and sufficient for the sufficiency of $\boldsymbol{p}$.

Thus the postulates of the theorem hold if and only if $\boldsymbol{p}^*$, the artificial economy price, is a sufficient statistic. The proof is completed by showing that $\boldsymbol{p}^*$ is a sufficient statistic if and only if there is a sufficient statistic equilibrium in the original model.

If the conditional distribution of $(z_i, \boldsymbol{r})$ given $(\boldsymbol{a}_i, \boldsymbol{p})$ is the same as the conditional distribution given complete information $\boldsymbol{a}$, dealers' trades given a realization $p$ of $\boldsymbol{p}$ are the same as if $\boldsymbol{p}^* = p$. Therefore if $\boldsymbol{p}$ is a market clearing price $\boldsymbol{p} = \boldsymbol{p}^*$. A rational expectations equilibrium exists in which dealers trade as if they had complete information if and only if the conditional distributions of $(z_i, \boldsymbol{r})$ given $(\boldsymbol{a}_i, \boldsymbol{p}^*)$ and $\boldsymbol{a}$ are the same $(i = 1, \ldots, n)$. $(z_i, \boldsymbol{r}, \boldsymbol{a})$ is a multivariate normal random variable and $(\boldsymbol{a}_i, \boldsymbol{p})$ is a linear function of $\boldsymbol{a}$ if and only if[9]

$$E(z_i, \boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}^*) = E(z_i, \boldsymbol{r} \mid \boldsymbol{a}). \tag{2.21}$$

If $\boldsymbol{p}^*$ is a sufficient statistic[10]

$$E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}^*) = E(\boldsymbol{r} \mid \boldsymbol{a}). \tag{2.22}$$

As $\boldsymbol{p}^*$ is a linear function of $\boldsymbol{a}$, Assumption 7 that

$$E(z_i \mid \boldsymbol{a}_i) = E(z_i \mid \boldsymbol{a}) \tag{2.23}$$

implies that

$$E(z_i \mid \boldsymbol{a}_i, \boldsymbol{p}^*) = E(z_i \mid \boldsymbol{a}). \tag{2.24}$$

Thus if $\boldsymbol{p}^*$ is a sufficient statistic for information about $\boldsymbol{r}$, $(\boldsymbol{a}_i, \boldsymbol{p}^*)$ is equivalent to complete information, and there is a rational expectations equilibrium with price $\boldsymbol{p}^*$. Conversely every rational expectations equilibrium in which the price is a sufficient statistic must have the same price as the unique artificial economy equilibrium. Thus there is no more than one sufficient statistic equilibrium.

### 2.3 *Proof of Theorem* 2

The proof makes extensive use of the following characterization of conditional expectations of normal random variables:

[9]This is another application of the argument of footnote 8.

[10]The derivations of (2.22) and (2.24) both use the fact that if $(x, \boldsymbol{a})$ is a multivariate normal random variable, $\boldsymbol{b}(\boldsymbol{a})$ and $\boldsymbol{c}(\boldsymbol{a})$ are linear functions of $\boldsymbol{a}$, and

$$E(x \mid \boldsymbol{b}(\boldsymbol{a})) = E(x \mid \boldsymbol{a}),$$

then

$$E(x \mid \boldsymbol{b}(\boldsymbol{a}), \boldsymbol{c}(\boldsymbol{a})) = E(x \mid \boldsymbol{a}).$$

The simplest proof of this uses the projection characterization (equations (2.25) and (2.26)) introduced at the beginning of the proof of Theorem 2: $E(x \mid \boldsymbol{b}(\boldsymbol{a}))$ is a linear function of $(\boldsymbol{b}(\boldsymbol{a}), \boldsymbol{c}(\boldsymbol{a}))$, $E(E(x \mid \boldsymbol{b}(\boldsymbol{a})) = Ex$, and as $E(x \mid \boldsymbol{b}(\boldsymbol{a})) = E(x \mid \boldsymbol{a})$, $x - E(x \mid \boldsymbol{b}(\boldsymbol{a}))$ is uncorrelated with $\boldsymbol{a}$ and thus with $(\boldsymbol{b}(\boldsymbol{a}), \boldsymbol{c}(\boldsymbol{a}))$. Thus $E(x \mid \boldsymbol{b}(\boldsymbol{a})) = E(x \mid \boldsymbol{b}(\boldsymbol{a}), \boldsymbol{c}(\boldsymbol{a}))$.

PROJECTION CHARACTERIZATION: If $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is a multivariate normal random variable $E(\boldsymbol{x}_1 \mid \boldsymbol{x}_2)$ is the *unique* linear function of $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ with the two following properties:

$$(2.25) \quad E(E(\boldsymbol{x}_1 \mid \boldsymbol{x}_2)) = E\boldsymbol{x}_1$$

and

$$(2.26) \quad \operatorname{cov}(\boldsymbol{x}_1 - E(\boldsymbol{x}_1 \mid \boldsymbol{x}_2), \boldsymbol{x}_2) = 0.$$

(2.25) is a property of all conditional expectations. (2.26) is equivalent to the statement that if $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is a zero-mean normal random variable, $E(\boldsymbol{x}_1 \mid \boldsymbol{x}_2)$ is the projection of $\boldsymbol{x}_1$ onto the subspace generated by the components of $\boldsymbol{x}_2$ in the Hilbert space of square integrable functions.[11] This generates a series of geometrical insights; the covariance of two random variables corresponds to the dot product in Euclidean space; uncorrelated random variables correspond to orthogonal vectors. These insights yielded the intuition on which the proof is based.

The great power of this characterization is that showing that a linear function of $\boldsymbol{x}_2$ satisfies (2.25) and (2.26) is sufficient to prove that it is the conditional expectation of $\boldsymbol{x}_1$ given $\boldsymbol{x}_2$. This implicit use of the projection theorem serves to eliminate a great deal of matrix algebra.

It is a matter of straightforward manipulation to show that (2.25) and (2.26) are equivalent to the standard formula when the variance matrix of $\boldsymbol{x}_2$ is nonsingular. The result is also true when the variance matrix is singular.[12]

The proof begins by assuming that there is a rational expectations equilibrium price $\boldsymbol{p}$ which is a linear function of $\boldsymbol{a}$. I will show that if the postulates of the theorem ((2.3)–(2.6)) hold, $\boldsymbol{p}$ must be a sufficient statistic, and so as I argued earlier $\boldsymbol{p} = \boldsymbol{p}^*$.

[11] See Loeve [**13**]. Projection techniques are familiar from regression (Luenberger [**14**]) where realizations of random variables are considered as elements in Euclidean space. Here the random variables themselves are considered as elements of function space.

[12] These formulae are: If

$$(\boldsymbol{x}_1, \boldsymbol{x}_2) \sim N\left[(\mu_1, \mu_2), \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right]$$

the conditional distribution of $\boldsymbol{x}_1$ given $\boldsymbol{x}_2 = x_2$ is normal with mean

$$E(\boldsymbol{x}_1 \mid \boldsymbol{x}_2 = x_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

and variance

$$\operatorname{var}(\boldsymbol{x}_1 \mid \boldsymbol{x}_2 = x_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

(See Anderson [**2**, Chapter 2].) If $\Sigma_{22}$ is singular it can be shown that (Anderson [**2**, page 24])

$$\boldsymbol{x}_2 = C\boldsymbol{x}_3 + \lambda$$

where $C$ has full column rank and $\operatorname{var} \boldsymbol{x}_3$ is nonsingular. As $C$ is of full column rank $\operatorname{cov}(\boldsymbol{x}_2, \boldsymbol{y}) = 0$ if and only if $\operatorname{cov}(\boldsymbol{x}_3, \boldsymbol{y}) = 0$. Thus (2.25) and (2.26) imply that $E(\boldsymbol{x}_1 \mid \boldsymbol{x}_2) = E(\boldsymbol{x}_1 \mid \boldsymbol{x}_3)$.

As $\boldsymbol{p}$ clears the market

$$(2.27)\qquad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}_i, \boldsymbol{p}) + \alpha_i \boldsymbol{p} - \beta_i E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) = 0$$

where $\alpha_i$ and $\beta_i$ are positive, nonstochastic functions of $\operatorname{var}(z_i, \boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p})$. From (2.3) and (2.6)

$$(2.28)\qquad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}) = \phi - \beta_0 E(\boldsymbol{r} \mid \boldsymbol{a}), \qquad \beta_0 \geqslant 0.$$

From Assumption 7,

$$(2.29)\qquad E(z_i \mid \boldsymbol{a}) = E(z_i \mid \boldsymbol{a}_i, \boldsymbol{p}).$$

Also as $\boldsymbol{p}$ is a function of $\boldsymbol{a}$,

$$(2.30)\qquad E(\boldsymbol{r} \mid \boldsymbol{a}) = E(\boldsymbol{r} \mid \boldsymbol{a}, \boldsymbol{p}).$$

From (2.28) and (2.29) and (2.30),

$$(2.31)\qquad \sum_{i=1}^{n} E(z_i \mid \boldsymbol{a}_i, \boldsymbol{p}) = \phi - \beta_0 E(\boldsymbol{r} \mid \boldsymbol{a}, \boldsymbol{p}).$$

From (2.31) and (2.27)[13]

$$\boldsymbol{p} = \psi + \lambda_0 E(\boldsymbol{r} \mid \boldsymbol{a}, \boldsymbol{p}) + \sum_{i=1}^{n} \lambda_i E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p})$$

or, letting $\boldsymbol{a}_0 = \boldsymbol{a}$,

$$(2.32)\qquad \boldsymbol{p} = \psi + \sum_{i=0}^{n} \lambda_i E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p})$$

where

$$(2.33)\qquad \psi = -\phi \Big/ \sum_{i=1}^{n} \alpha_i,$$

$$(2.34)\qquad \lambda_i = \beta_i \Big/ \sum_{i=1}^{n} \alpha_i \qquad\qquad (i = 0, 1 \ldots n).$$

By assumption (2.6) $\beta_0 \geqslant 0$. From (1.15) and (1.16) $\alpha_i > 0$, $\beta_i > 0$, $i = 1 \ldots n$. Thus

$$(2.35)\qquad \lambda_0 \geqslant 0, \qquad \lambda_i > 0 \qquad\qquad (i = 1, \ldots n).$$

[13]The argument which follows is conducted in terms of variances. It is equivalent to more abstract argument in which equation (2.32) is projected onto the subspace of Hilbert space, generated by linear combinations of the components of $\boldsymbol{a}$ which are orthogonal to $\boldsymbol{p}$. The left-hand side becomes zero; the right-hand side is the sum of terms, all of which are nonnegatively correlated with $\boldsymbol{r}$, which must therefore all be zero.

(2.32) and (2.35) are the crucial equations from which the rest of the proof follows. From (2.32) as $\lambda_i$, $i = 0, \ldots n$, and $\psi$ are not random variables,

$$(2.36) \quad \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), \boldsymbol{p}) = \sum_{i=0}^{n} \lambda_i \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p})).$$

But from the 'projection characterization' (2.26)

$$(2.37) \quad \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), \boldsymbol{p}) = 0.$$

Also

$$(2.38) \quad \begin{aligned} &\operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p})) \\ &= \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})) \\ &\quad + \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{p})) \\ &= \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})). \end{aligned}$$

((2.26) and the fact that $E(\boldsymbol{r} \mid \boldsymbol{p})$ is a linear function of $\boldsymbol{p}$ imply that $\operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{p})) = 0$.) But

$$(2.39) \quad \begin{aligned} &\operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})) \\ &= \operatorname{var}(E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})) \\ &\quad + \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})) \\ &= \operatorname{var}(E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})) \end{aligned}$$

(as $\operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}), (\boldsymbol{a}_i, \boldsymbol{p})) = 0$ and $E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})$ is a linear function of $(\boldsymbol{a}_i, \boldsymbol{p})$). From (2.38) and (2.39)

$$(2.40) \quad \operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p})) = \operatorname{var}(E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})) \geqslant 0 \qquad (i = 0, \ldots n).$$

The conditional expectation, $E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p})$, cannot be negatively correlated with $\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p})$. From (2.36), (2.37), and (2.40),

$$(2.41) \quad 0 = \sum_{i=0}^{n} \lambda_i \operatorname{var}(E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) - E(\boldsymbol{r} \mid \boldsymbol{p})).$$

As $\lambda_0 \geqslant 0$ and $\lambda_i > 0$ $(i = 1, \ldots n)$ (2.41) implies that

$$(2.42) \quad E(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) = E(\boldsymbol{r} \mid \boldsymbol{p}) \qquad (i = 1, \ldots n).$$

From (2.42) and the projection characterization

$$\operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), \boldsymbol{a}_i) = 0 \qquad (i = 1, \ldots n).$$

Thus $\operatorname{cov}(\boldsymbol{r} - E(\boldsymbol{r} \mid \boldsymbol{p}), \boldsymbol{a}) = 0$ since $\boldsymbol{a} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$. $E(\boldsymbol{r} \mid \boldsymbol{p})$ is a linear function of $\boldsymbol{p}$ and thus of $\boldsymbol{a}$, and $E(E(\boldsymbol{r} \mid \boldsymbol{p})) = E\boldsymbol{r}$. Thus $E(\boldsymbol{r} \mid \boldsymbol{p}) = E(\boldsymbol{r} \mid \boldsymbol{a})$. $\boldsymbol{p}$ is a sufficient statistic, and thus as $\boldsymbol{p}$ is also a market clearing price $\boldsymbol{p} = \boldsymbol{p}^*$.

## 3. DISCUSSION

In this model the price does not, in general, provide information which is equivalent to complete information. This contrasts with the work of other authors (Allen [1], Grossman [7, 8, 9], Jordan [11], Kihlstrom and Mirman [12], and Radner [17].[14] The difference between this and other models, which accounts for the different results, is my assumption that some of the agents in this model are producers, who have random second period endowments ($z_i$) which are correlated with asset returns. The random endowments here are physical output; but there are a number of other situations in which people obtain random income from sources other than their portfolio of financial assets (for example, labor income, gifts, and bequests). The models of Grossman and Stiglitz [10] and Danthine [4] also include random endowments, but differ from this model in making the endowments the sole source of uncertainty about which dealers have any information.

Theorem 1 establishes that in this model, if there are two sources of uncertainty, one of which is endowments, the futures price is not in general a sufficient statistic. This is established by using Grossman's concept of an artificial economy in which dealers pool all their information prior to trading. There is a rational expectations equilibrium in which prices are a sufficient statistic if and only if the artificial economy prices are a sufficient statistic. If dealers' trades in the artificial economy are affected by information about both asset returns and endowments the artificial economy price reflects both types of information, and cannot in general be a sufficient statistic. This observation is similar to Grossman's [7] comment that introducing 'noise' in the form of random supply into a stock market model prevents the price from being a sufficient statistic. However it is not clear that 'noise' is the right word for information about endowments which are correlated with asset return; in the absence of other information the 'noise' becomes the signal. A better way of describing the situation is that the artificial economy price is a function of two signals, one about asset returns, the other about random endowments which themselves affect asset returns; it is usually impossible to disentangle the effects of these signals.[15]

[14] With the exception of Jordan [11] the major thrust of these papers is to establish conditions under which prices reveal information which is equivalent to perfect information. Jordan takes the opposite approach, establishing that if the number of markets is too small relative to the dimension of the available information space the price cannot reveal all the information, except when certain conditions hold of the utility function. One of these is constant absolute risk aversion which is assumed here.

This model is not a special case of Jordan's, fundamentally because it allows for random endowments, which are correlated with returns. Jordan's model also precludes normal random variables, by assuming bounded supports.

[15] In examples of this type if sufficient additional markets are opened in which trade is a function of the same information variables, the dimension of the price vector becomes high enough for it to be a sufficient statistic. Grossman [8] contains an example in which the opening of a futures market has this effect.

Allen [1] and Jordan [11] have more general results relating to the interaction of the dimension of the information space and the number of markets. Grossman [9] shows that if the information and return variables are normal, there is a riskless asset, and the market portfolio of risky assets is not a Giffen good; the dimension of the information space is effectively the number of risky assets, regardless of how many information variables there are, and prices are a sufficient statistic.

Another effect of the random endowments (output) is the absence of the paradox discussed by Grossman [7] and Grossman and Stiglitz [**10**]. The market price can only reflect the information which dealers collect. If no information is collected the price is uninformative so dealers have an incentive to collect costly information; however, if they collect the information, it is all revealed costlessly by prices so there is no incentive to collect it. This model avoids this paradox in two ways. Firstly, price may not be a sufficient statistic. Secondly, if the price is a sufficient statistic, it aggregates information about the spot price $\boldsymbol{r}$ so

$$E(\boldsymbol{r} \mid \boldsymbol{a}) = E(\boldsymbol{r} \mid \boldsymbol{p}). \tag{3.1}$$

(3.1) does not imply that

$$E(z_i \mid \boldsymbol{a}) = E(z_i \mid \boldsymbol{p}) \qquad (i = 1, \ldots n). \tag{3.2}$$

The futures price is not a sufficient statistic for information about each dealer's output. Producers may still wish to gather information which tells them nothing about the spot price $\boldsymbol{r}$ which they cannot already infer from the futures price, because the information tells them about their own output. The futures market aggregates this information—and so the futures price is informative about the spot price. There are still incentives to gather information about output even when the futures price is a sufficient statistic for information about the spot price.

The second theorem is concerned with situations in which there is a rational expectations equilibrium in which the price is a sufficient statistic. In these circumstances (from (2.3))

$$E\left(\sum_{i=1}^{n} z_i \mid \boldsymbol{a}\right) = \phi - \beta_0 E(\boldsymbol{r} \mid \boldsymbol{a}). \tag{3.3}$$

I establish that if $\beta_0$ is nonnegative there is no other rational expectations equilibria in which the price is a linear function of $\boldsymbol{a}$. When there is information about only one side of the spot market, (3.3) holds with $\beta_0 \geqslant 0$. If there is information about both sides of the spot market (3.3) is unlikely to hold; so the theorem covers the most economically plausible cases of a sufficient statistic equilibrium.

However the theorem has serious limitations in dealing with the possibility of equilibria in which the price is a nonlinear function of the information variables. There can be no equilibria in which the price is a nonlinear invertible function of a linear function of information, because then the conditional distribution of $(z_i, \boldsymbol{r})$ given $(\boldsymbol{a}_i, \boldsymbol{p})$ is normal; demand is a linear function of price and information, and so the market clearing price must be a linear function of the information. However there might be other equilibria with nonlinear prices. The conditional distribution of $(z_i, \boldsymbol{r})$ in these equilibria would not be normal, and the methods of this paper are not applicable.

Nevertheless the theorem does eliminate a large class of potential equilibria from consideration. It is proved by an argument which is equivalent to use of the projection theorem, which seems to be a powerful tool for the analysis of linear rational expectations models.[16]

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## APPENDIX

### CALCULATION OF DEALERS' FUTURES SALES

The objective here is to find $y_i$ which maximizes

$$V_i = E(-\exp(-k_i \boldsymbol{R}_i) \mid \boldsymbol{a}_i = a_i, \boldsymbol{p} = p)$$

where $\boldsymbol{R}_i = y_i(p - \boldsymbol{r}) + z_i \boldsymbol{r}$. It is convenient to drop the subscript $i$ and write

$$(\bar{z}, \bar{r}) = E(z, \boldsymbol{r} \mid \boldsymbol{a} = a, \boldsymbol{p} = p)$$

and

$$\Sigma = \begin{bmatrix} \sigma^2 & \rho \\ \rho & \omega^2 \end{bmatrix} = \operatorname{var}(z, \boldsymbol{r} \mid \boldsymbol{a} = a, \boldsymbol{p} = p).$$

As $(z, \boldsymbol{r}, \boldsymbol{a}, \boldsymbol{p})$ is multivariate normal, the conditional distribution of $(z, \boldsymbol{r})$ given $(\boldsymbol{a}, \boldsymbol{p})$ is bivariate normal, $\Sigma$ does not depend on the realization $(a, p)$, and $\Sigma$ is positive semi-definite. (See Anderson [**2**, page 29].)

In one situation the calculation is straightforward. If $\sigma^2 = \rho = 0$, $\boldsymbol{R}$ is normal;

$$E(\boldsymbol{R} \mid \boldsymbol{a} = a, \boldsymbol{p} = p) = y(p - \bar{r}) + \bar{z}\bar{r},$$

$$\operatorname{var}(\boldsymbol{R} \mid \boldsymbol{a} = a, \boldsymbol{p} = p) = (\bar{z} - y)^2 \omega^2.$$

The moment generating function of $\boldsymbol{x} \sim N(\mu, \sigma^2)$ is

$$m(t) = E(\exp(t\boldsymbol{x})) = \exp\left(t\mu + \tfrac{1}{2}\sigma^2 t^2\right)$$

(Mood and Graybill [**15**, p. 126]).

Therefore, in this case

$$V = E(-\exp(-k\boldsymbol{R}) \mid \boldsymbol{a}, \boldsymbol{p}),$$

$$V = -m(-k) = -\exp\left(-k(y(p - \bar{r}) + \bar{z}\bar{r}) + \tfrac{1}{2}k^2(\bar{z} - y)^2\omega^2\right).$$

From (1.13) $\omega^2 = \operatorname{var}(\boldsymbol{r} \mid \boldsymbol{a}_i, \boldsymbol{p}) > 0$ so $V$ is maximized when

$$y = \bar{z} + (1/k\omega^2)(p - \bar{r})$$
$$= \bar{z} + \alpha p - \beta \bar{r}$$

where

$$\alpha = \left[(1 + k\rho)^2 - k^2\sigma^2\omega^2\right]/k\omega^2,$$

$$\beta = (1 + k\rho)/k\omega^2,$$

[16] See also Futia [**6**].

as

$$\sigma^2 = \rho = 0.$$

If $\sigma^2 = \text{var}(z_i \mid \boldsymbol{a}_i, \boldsymbol{p}) > 0$, $\boldsymbol{R}$ includes the nonnormal term $\boldsymbol{zr}$ so is not itself normal, and does not have a standard moment generating function. Here calculation from first principles is a tedious necessity, and conditions have to be imposed to ensure that the utility integral is finite. I give the details for the case where $\Sigma$ is nonsingular: In this case,

$$V = -2\Pi |\Sigma|^{-1/2} \int_{r=-\infty}^{\infty} \int_{z=-\infty}^{\infty} \exp(-W)\, dz\, dr,$$

where

$$W = kR + \tfrac{1}{2}(z - \bar{z}, r - \bar{r})\Sigma^{-1}(z - \bar{z}, r - \bar{r})'.$$

The first term in $W$ arises from the utility function, the second from the normal density function. Rearranging,

$$W = \tfrac{1}{2}x'Ax + b'x + c,$$

where

$$x = (z - \bar{z}, r - \bar{r})',$$

$$A = (\sigma^2\omega^2 - \rho^2)^{-1} \begin{bmatrix} \omega^2 & \omega^2(\beta - a) \\ \omega^2(\beta - \alpha) & \sigma^2 \end{bmatrix},$$

$$\alpha = \left[(1 + k\rho)^2 - k^2\sigma^2\omega^2\right]/k\omega^2,$$

$$\beta = (1 + k\rho)/k\omega^2,$$

$$b = k(\bar{r}, \bar{z} - y)',$$

and

$$c = k(\bar{z}\bar{r} + y(p - \bar{r})).$$

$V$ is finite if and only if $A$ is positive definite. I now show that in this case $\alpha$ and $\beta$ are positive, and that Assumption 8 is sufficient for this.

$$\det A = k\omega^2\alpha/(\sigma^2\omega^2 - \rho^2) > 0 \quad \text{if and only if} \quad \alpha > 0.$$

But

$$\alpha = [1 + k(\sigma\omega + \rho)][1 - k(\sigma\omega - \rho)]/k\omega^2.$$

As $k(\sigma\omega + \rho) > 0$, $\alpha > 0$ if and only if $1 - k(\sigma w - \rho) > 0$, in which case

$$\beta = \frac{1 + k\rho}{k\omega^2} > \sigma/\omega > 0,$$

so $\beta$ is also positive. Now $\rho \geqslant -\sigma w$ so

$$1 - k(\sigma w - \rho) \geqslant 1 - 2k\sigma\omega;$$

so it is sufficient for $\alpha > 0$ that

$$1 > 4k^2\sigma^2\omega^2.$$

But

$$\sigma^2\omega^2 = \operatorname{var}(z \mid \boldsymbol{a}, \boldsymbol{p})\operatorname{var}(\boldsymbol{r} \mid \boldsymbol{a}, \boldsymbol{p}) \leqslant \operatorname{var} z \operatorname{var} \boldsymbol{r}.$$

So a sufficient condition is

$$4k^2 \operatorname{var} \boldsymbol{r} \operatorname{var} z < 1$$

or as $\boldsymbol{r} = (1/\delta)(e - \sum_{i=1}^{n} z_i)$ (from (1.10))

$$(4k_i^2/\delta^2)\operatorname{var}\left(e - \sum_{j=1}^{n} z_j\right)\operatorname{var} z_i < 1 \qquad (i = 1, \ldots n).$$

This is Assumption 8.

Returning to the calculation of expected utility, on the assumption that $A$ is positive definite,

$$A = BB'.$$

Writing $u = B'x + B^{-1}b$,

$$\int_{r=-\infty}^{\infty}\int_{z=-\infty}^{\infty} \exp(-W)\,dz\,dr = \int_{\mathbb{R}^2} \exp\left(-\tfrac{1}{2}x'Ax - b'x - c\right)dx$$

$$= |\det A|^{-1/2}\int_{\mathbb{R}^2} \exp\left(-\tfrac{1}{2}u'u + \tfrac{1}{2}b'A^{-1}b - c\right)du$$

$$= 2\Pi\,|\det A|^{-1/2}\exp\left(\tfrac{1}{2}b'A^{-1}b - c\right).$$

Thus

$$V = -|\Sigma|^{-1/2}|\det A|^{-1/2}\exp\left(\tfrac{1}{2}b'A^{-1}b - c\right).$$

$\Sigma$ and $A$ do not involve $y$, but $b$ and $c$ do. Thus $V$ is maximized when

$$\tfrac{1}{2}b'A^{-1}b - c$$

is minimized, that is, when

$$\left(\frac{db}{dy}\right)'A^{-1}b - \frac{dc}{dy} = 0$$

or

$$y = z + \alpha p - \beta\bar{r}.$$

## REFERENCES


[1] Allen, B.: "Generic Existence of Completely Revealing Equilibria for Economies with Uncertainty When Prices Convey Information," forthcoming in *Econometrica*.

[2] Anderson, T. W.: *An Introduction to Multivariate Statistical Analysis*. New York: John Wiley and Sons, 1958.

[3] Bray, M. M.: "Information in Futures Markets," Bachelor of Philosophy Thesis, University of Oxford, 1978.

[4] Danthine, J. P.: "Information, Futures Prices and Stabilizing Speculation," *Journal of Economic Theory*, 17 (1978), 79–98.

[5] Fama, E. F.: "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance*, 25 (1970), 383–416.

[6] Futia, C. A.: "Rational Expectations in Linear Models," *Econometrica*, 49 (1981), 171–192.
[7] Grossman, S. J.: "On the Efficiency of Competitive Stock Markets Where Traders Have Diverse Information," *Journal of Finance*, 31 (1976), 573–585.
[8] ———: "The Existence of Futures Markets, Noisy Rational Expectations and Informational Externalities," *Review of Economic Studies*, 44 (1977), 431–449.
[9] ———: "Further Results On the Informational Efficiency of Competitive Stock Markets," *Journal of Economic Theory*, 18 (1978), 81–101.
[10] Grossman, S. J., and J. E. Stiglitz: "Information and Competitive Price Systems," *American Economic Review*, 66 (1976), 246–253.
[11] Jordan, J. S.: "On the Efficient Markets Hypothesis," mimeograph, University of Minnesota, 1979.
[12] Kihlstrom, R., and L. J. Mirman: "Information and Market Equilibrium," *Bell Journal of Economics and Management Science*, 6 (1975), 357–376.
[13] Loeve, M.: *Probability Theory*. Princeton: Van Nostrand, 1955.
[14] Luenberger, D. G.: "Optimization by Vector Space Methods," New York: John Wiley and Sons, 1969.
[15] Mood, A. M., and F. A. Graybill: *Introduction to the Theory of Statistics*. New York: McGraw-Hill, 1963.
[16] Muth, J. F.: "Rational Expectations and the Theory of Price Movements," *Econometrica*, 9 (1961), 315–335.
[17] Radner, R.: "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices," *Econometrica*, 47 (1979), 655–678.
[18] Samuelson, P. A.: "Proof that Properly Anticipated Prices Fluctuate Randomly," in *Collected Scientific Papers of P. A. Samuelson*, Vol. III. Cambridge: MIT Press, 1972, pp. 782–790.
[19] ———: "Proof that Properly Discounted Present Values of Assets Vibrate Randomly," *Bell Journal of Economics and Management Science*, 4 (1973), 369–374.