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# Stock Prices:

## Random vs. Systematic Changes

The subject matter of this paper is bound to be considered heresy. I can say that without equivocation, because whatever views anyone expresses on this subject are sure to conflict with someone else's deeply-held beliefs.[1]

For the purpose of exposition, I can characterize these beliefs as falling into two classes. Apparently there are, or were, a substantial group of economists who believe(d) that common stock prices tended to move in a deterministic, cyclical manner, where the term "cyclical" is taken, not in the sense that the National Bureau uses it, but in the mechanical sense of a movement perfectly predictable in timing and extent. These cycles might be quite complex, but diligent effort, and perhaps Fourier analysis, would eventually solve the riddle and bring a pot of gold to the persevering.

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[1] There have been a considerable number of articles dealing with the subject of independence in price changes in speculative markets over the last two decades. The earliest work on this subject was by a French mathematician statistician, M. L. Bachelier, Ref. (2). H. Working investigated the subject in Ref. (21). A. Cowles and H. Jomes did further work in 1937, Ref. (6). H. Working followed with Ref. (23), and M. G. Kendall examined the subject in Ref. (13). More recent investigations have been made by M. F. M. Osborne, Ref. (17); H. Houthakker, Ref. (10); S. Alexander, Ref. (1); A. Larsen, Ref. (15); H. Working, Ref. (22); A. Cowles, Ref. (5); and H. Roberts, Ref. (18).

Related to this question is the recent extensive controversy about risk premiums for which a bibliography is available in P. H. Cootner, Ref. (4), P. H. Cootner, "Risk Premiums: Seek and Ye Shall Find" (forthcoming).

There are also a number of unpublished sources on the subject: a paper by Paul A. Samuelson on Brownian motion in the stock market; a Ph.D. thesis by Arnold Moore at the University of Chicago; a bachelor's thesis by R. Cryer at M.I.T.; and related work in Ph.D. theses on put, call, and warrant markets by Richard Kruizenga at M.I.T., Case Sprenkle at Yale and James Boness at Chicago. Other work is in progress on commodity markets at the Food Research Institute of Stanford University, principally by H. Working, and at Cornell University under the direction of S. Smidt and M. DeChazeau.

Except for a small fringe of opinion largely confined to stock market professionals, this point-of-view is moribund today, but its refutation has bred an important and intellectually intriguing countertheory.[2] The stock exchange is a well-organized, highly-competitive market. Assume that, in fact, it is a perfect market. While individual buyers or sellers may act in ignorance, taken as a whole, the prices set in the marketplace thoroughly reflect the best evaluation of currently available knowledge. If any substantial group of buyers thought prices were too low, their buying would force up the prices. The reverse would be true for sellers. Except for appreciation due to earnings retention, the conditional expectation of tomorrow's price, given today's price, is today's price.

In such a world, the only price changes that would occur are those which result from new information. Since there is no reason to expect that information to be non-random in its appearance, the period-to-period price changes of a stock should be random movements, statistically independent of one another. The level of stock prices will, under these conditions, describe what statisticians call a random walk, and physicists call Brownian motion. In the normal course of events, the level of prices, i.e., the summation of these random movements, will show movements that look like cycles but in fact are not.[3] Nothing can be learned about the future from looking at these price series. Buying a stock based on signals from such a chart will produce results no better than those from repeated flipping of a fair coin. The time might just as well be spent on analyzing the results of a fair roulette wheel.

You can see why the idea is intriguing. Where else can the economist find that ideal of his - the perfect market? Here is a place to take a stand, if there is such a place.

Unfortunately, it is not the right place. The stock market is not a random walk. A growing number of investigators have begun to suspect it,[4] and I think I have enough evidence to demonstrate the nature of the imperfections. On the other hand, I do not believe that the market is grossly imperfect. In fact, I do not know why the process I see going on in the market is not worthy of the name perfection too. It strays from "perfection" only to the extent that it defines the Frank Knight - Milton

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[2] A more complete exposition can be found in H. Roberts, Ref. (18), or H. Working, Ref. (23).

[3] For discussions of this phenomena, see W. Feller, Ref. (8), and J. E. Kerrich, Ref. (14).

[4] Principally Houthakker, Alexander, Moore, and Larsen, though it should be pointed out that Moore and Larsen prefer to stress the randomness rather than the imperfections.

Friedman assumption of profitless speculation. Even more interesting, perhaps, is that my model is perfectly compatible with much of what I interpret Wall Street chart reading to be all about. Like the Indian folk doctors who discovered tranquilizers, the Wall Street witch doctors, without the benefit of scientific method, have produced something with their magic, even if they can't tell you what it is or how it works.

In this paper I will present a hypothesis which fits the data much better, and which has implications substantially different from that of the random walk hypothesis. There is a certain tentativeness about these results, however, because the testing is not quite complete. For one thing, although the basic outlines of the hypothesis I will present here were formulated in advance of the testing, some modifications were made in the course of the testing, so it is not truly appropriate to consider the results to be a proof of the theory. Secondly, while the hypothesis was tested against a wide variety of stock prices, the stock sample was not randomly drawn. Both of these deficiencies will be eliminated in further testing now under way, but for the time being these results are only tentative.

## THE MODEL

First, I will present my own model of stock market behavior along with its statistical implications. After that I will compare the results of some statistical analysis with the implications of the two competing models. Finally, I will outline the further work contemplated along these lines and their implications for tests of randomness.

Let us start with the concept of a perfect market held by random walk theorists, but let us achieve this perfection without assuming a high degree of knowledgeability of the participants. They are all engaged in other occupations in which they have a comparative advantage so it is very costly, at least in terms of opportunity cost per unit of relevant information uncovered, for them to devote time to the relevant kind of stock market research. (In Stiglerian terms, their cost of search is very high.) [5] As a result, they tend to accept present prices as roughly representing true differences in value and they choose between stocks largely on the grounds of their attitudes toward risk. Those of them that do choose among stocks on the basis of information about future prospects are just as likely to be wrong as not. Demand for stocks will mostly depend upon changes in the level of income and its distribution among stockholders with different preferences. As the present moves into the future, the stockholders will face all kinds of surprises, but most of the surprises which come this week will not be related, in any systematic manner, to the surprises which will come next week.

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[5] G. Stigler, Ref. (20).

In R. Nelson's terms, they have a prediction equation with a larger $R^2$ (correlation coefficient) and so the value of a piece of information is greater. Ref. (16).

Now let me introduce another group of investors and speculators who specialize in the stock market. As professionals, their opportunity cost of research is much less than that of the uninformed (largely because they know what to look for and where), but it is, nevertheless, non-negative. They do have an idea of what is going to happen in the future, but they cannot profit from it unless the current price deviates enough from the expected price to cover their opportunity costs. Their profits will come from observing the random walk of the stock market prices produced by the non-professionals until the price wanders sufficiently far from the expected price to warrant the prospect of an adequate return. In other words, when prices have deviated enough from the expected price that they can expect future surprises to force prices toward their mean more often than not. Competition among these professionals will tend to restrict the potential profit to the opportunity costs. Furthermore, they must recognize the possibility of error in their own forecasts and must recognize that even if they buy the stock at a favorable price the actual rate at which the stock appreciates or depreciates will be governed in fact by the random rate of approach to the expected price.

Let me illustrate. For simplicity, assume that every professional has the same expectations, and the same opportunity costs. Then prices will behave as a random walk with reflecting barriers. In something like the manner once suggested by Taussig, prices within those upper and lower limits will tend to move like a random walk. If prices fell to the lower limit, however, the rate at which the price moves back to the expected price is governed by the random process which operates within the barriers, so that even if their expectations are correct, their profit rate is still a stochastic variable. In addition, the individual buyer is unlikely to know for certain that his estimate of the price is correct or that other professionals share his estimate.

Furthermore, within this class of professionals, another sort of random walk environment operates. There is no reason to expect that changes in the price expectations of professionals should occur in other than a random manner. As a result, there probably would also be random changes in the trends around which the random walk takes place. That is, the path of stock prices over any substantial period of time would be composed of a random number of trends, each of which is a random walk with reflecting barriers. There is much random behavior in such a series, but it is substantially different from a random walk, and while it has some implications which are quite similar to that of a random walk, it has others which are strikingly different, as we shall see before long.

Note, for example, the customary distribution of weekly changes in the logarithms of stock prices. The mean of such weekly changes is very likely to be less than 0.005; the standard deviation, on the other hand, is likely to be much larger, usually between 0.02 and 0.03. (For these purposes, we will accept this fact as given, although it could be developed from the general theory.) (Table I.) One implication of this, of course, is that it would be very difficult to detect the significance of any weekly mean price change if the series were truly a random walk. But the high variance also means that any professional who did feel knowledgeable about the mean price would still want to set his buying price considerably away from the mean to protect against risks. If the lower edge of the barrier were to average several weekly standard deviations away from the mean, by far the greater portion of the successive weekly price changes would be totally uncorrelated with each other. On the other hand, when prices neared the barrier there would be a tendency for some negative autocorrelation, since movements to the barrier would be more likely to be followed by movements in the opposite direction. The net effect would be a moderate negative correlation near the boundary which would be heavily diluted by the number of cases when the price was near the mean which would show no such negative autocorrelation.[6]

In addition to this negative correlation, the effect of the barriers would be to produce more small price changes than would be expected from a normal distribution of price changes. When unencumbered by the barrier, the central limit theorem would tend to ensure that the total weekly effect of a large number of individual transaction price changes would be approximately normal. The existence of the barrier, however, would cut short some of the price movements toward the barriers without restricting as much the very large price movements which could still occur in the direction away from the barriers. We would expect the distribution of price changes over short periods of time to be more leptokurtic under such conditions than the normal distribution.

As we look at changes over longer periods of time, however, other factors, which are relatively unimportant in the case of weekly changes, become much more significant. I have spoken of stock price series as composed of several trends of different slopes. As we lengthen the period over which we take differences the mean becomes more important relative to the standard deviation. (In a random walk this would be true because the increase in the mean of the price changes is directly proportional to the interval over which the price change is measured, while the standard deviation increases only as the square root of the interval. If there are reflecting barriers, the standard deviation will increase less rapidly than in a random walk so this effect will be even more pronounced.) Furthermore, the mean of each of the component trends becomes more distinguishable from the group mean. This will result in an increasing element of positive autocorrelation as long as the time interval of the

[6] These, and similar results discussed later in the paper can readily be derived from basic texts in Markov chain theory. On this point, e.g., see Kemeny and Snell, Ref. (11).

TABLE I

| Company number | Company | Dates covered | Mean | Standard deviation |
|---|---|---|---|---|
| 1 | Socony Mobil | 56-60 | -0.0001 | 0.0298 |
| 2 | Westinghouse | 56-60 | 0.0025 | 0.0249 |
| 3 | Chrysler | 56-60 | -0.0020 | 0.0297 |
| 4 | Procter and Gamble | 56-60 | 0.0042 | 0.0226 |
| 5 | General Motors | 56-60 | 0.0004 | 0.0208 |
| 6 | RCA | 56-60 | 0.0011 | 0.0310 |
| 7 | Goodyear | 56-60 | 0.0026 | 0.0257 |
| 8 | General Foods | 56-60 | 0.0046 | 0.0258 |
| 9 | Commercial Solvents | 56-60 | 0.0013 | 0.0355 |
| 10 | B. F. Goodrich | 56-60 | -0.0008 | 0.0300 |
| 11 | du Pont | 56-60 | -0.0001 | 0.0195 |
| 12 | May Department Stores | 56-60 | 0.0010 | 0.0206 |
| 13 | Standard of New Jersey | 56-60 | 0.0000 | 0.0215 |
| 14 | Brunswick Corporation | 56-60 | 0.0130 | 0.0452 |
| 15 | Ford Motors | 56-60 | 0.0011 | 0.0257 |
| 16 | Douglas Aircraft | 56-60 | -0.0025 | 0.0281 |
| 17 | North American Aviation | 56-60 | 0.0013 | 0.0382 |
| 18 | Boeing Aircraft | 56-60 | 0.0012 | 0.0392 |
| 19 | International Paper | 56-60 | 0.0003 | 0.0230 |
| 20 | American Can | 56-60 | 0.0000 | 0.0190 |
| 21 | Allegheny Steel | 56-60 | 0.0013 | 0.0372 |
| 22 | Bethlehem Steel | 56-60 | 0.0008 | 0.0226 |
| 23 | Byers Corporation | 56-60 | -0.0006 | 0.0618 |
| 24 | Carpenter Steel | 56-60 | 0.0030 | 0.0383 |
| 25 | Colorado Fuel and Iron | 56-60 | -0.0014 | 0.0334 |
| 26 | Continental Steel | 56-60 | 0.0033 | 0.0465 |
| 27 | Granite City Steel | 56-60 | 0.0033 | 0.0355 |
| 28 | Inland Steel | 56-60 | 0.0020 | 0.0269 |
| 29 | Interlake Iron | 56-60 | -0.0002 | 0.0386 |
| 30 | Jones and Laughlin | 56-60 | 0.0014 | 0.0320 |
| 31 | Pittsburgh Steel | 56-60 | -0.0025 | 0.0665 |
| 32 | Republic Steel | 56-60 | 0.0013 | 0.0274 |
| 33 | U. S. Steel | 56-60 | 0.0019 | 0.0242 |
| 34 | Wheeling Steel | 56-60 | 0.0005 | 0.0288 |
| 35 | Conoco | 51-55 | 0.0041 | 0.0207 |
| 36 | Studebaker-Packard | 54-60 | -0.0017 | 0.0648 |
| 37 | Dow Chemical | 55-60 | 0.0022 | 0.0264 |
| 38 | Coca Cola | 56-60 | 0.0028 | 0.0221 |
| 39 | Nabisco | 50-60 | 0.0015 | 0.0164 |
| 40 | IBM | 50-60 | 0.0049 | 0.0277 |
| 41 | Reynolds Metals | 50-60 | 0.0056 | 0.0406 |
| 42 | Int'l. Telephone and Telegraph | 50-60 | 0.0004 | 0.0349 |
| 43 | Brown and Bigelow | 56-60 | 0.0022 | 0.0204 |
| 44 | Interstate Department Stores | 56-60 | 0.0025 | 0.0282 |
| 45 | National Dairy Products | 50-60 | 0.0031 | 0.0234 |

differencing is less than the length of the trends. That is, the successive changes in an uptrend will all tend to be higher than the overall mean and the successive changes in a downtrend will all tend to be lower.[7] As the differencing interval exceeds the length of some of the trends, this positive autocorrelation will begin to disappear.

The positive autocorrelation is also present in changes over one-week periods, but over such short periods the absolute magnitude of the differences in the means is so small relative to the standard deviation that the effect is negligible. It simply becomes increasingly prominent particularly when measured against the negative serial correlation induced by the barriers as the time interval increases.

In addition, the kurtosis of the distribution over longer differencing intervals will also change if the reflecting barrier hypothesis is true. If there were only a single trend, the distribution of price changes over successively longer time intervals will approach that of a rectangular distribution: i.e., it will be equally likely to get any value for the price change equal to or less than the width of the barriers. If the series were a single random walk, the distribution of price changes over successively longer intervals should become more and more normal as the central limit theorem becomes more and more applicable. So, if the random walk hypothesis is correct, kurtosis should be near 3 at weekly intervals and get closer to 3 as time goes on. If the reflecting barrier or trend hypothesis is correct, kurtosis should be greater than 3 to begin with and should approach the kurtosis of the rectangular distribution in the limit if a single trend is involved.

Where trends change over time, the predicted pattern is not completely clear, but the average value of the stock price <u>trend</u> over any substantial period of time is likely to be severely limited by the possibility of transferring funds among different investment outlets. Thus it is likely, under the reflecting barrier hypothesis, that kurtosis of the price changes will be somewhat larger than that of a rectangular distribution.

The primary element underlying all of these implications is that the stock price series will simply not be free to wander as much as they would if the series were a random walk. This tendency can be tested directly by utilizing the distributions of the range of a random walk developed by W. Feller.[8] Given these distributions it is possible to simply

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[7] This would be true even if each series was composed of several random walks each with a different mean. In this latter case, however, the positive autocorrelation would be much slower to appear.

[8] W. Feller, Ref. (9). M. Solari and A. Anis, Ref. (19), have shown that the mean of the actual distribution converges very slowly to the asymptotic distribution. It can be shown, however, that this slow convergence of the mean is due to the (small) probability of very large ranges under asymptotic conditions. Actually, the left-hand tail is a very close approximation to the exact distribution and it is this tail in which we will be interested.

calculate the probability that a segment of the series is drawn from a random walk. If my hypothesis is correct, it should be possible to break each series up into (a relatively few, long) segments each of which is significantly different from a random walk. The results of this test are discussed below.

I have drawn this hypothesis in fairly abstract terms for the sake of clarity,[9] but it is possible to relax many of the assumptions without substantially altering the conclusions. Instead of breaking investors into two categories, one almost completely uninformed, the other almost completely knowledgeable, all I need is that there should be a considerable gulf in knowledge between the two substantial groups of investors and that there are relatively few people in the penumbra. I think the principle of comparative advantage insures this division to the necessary degree.

Similarly, it is not necessary to assume that all professionals share exactly the same expectations or have the same search costs, so that the barriers to price change may be soft and rubberlike rather than

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[9] Statistically, the difference between this hypothesis and the simple random walk can be stated as follows. The general first-order autoregressive structure is of the form

$$X_{n+1} = a_n X_n + \beta + \sigma Y$$

where Y is a random variable from a normal density function with zero mean and unit variance. If the process is a random walk, $\beta$ is the mean price change per period and $a_n$ equals unity for all n. This reduces to

$$(X_{n+1} - X_n) = \beta + \sigma Y_n$$

i.e., the changes are distributed normally with mean $\beta$ and standard deviation $\sigma$. One possible deviation from randomness might be $a_n \neq 1$. In that case

$$X_{n+1} - X_n = (a_n - 1)X_n + \beta + \sigma Y$$

in which case the distribution of the changes would be autocorrelated. If $a_n > 1$ an increase in $X_n$ would raise the expectation that the next change would also be an increase. If $a_n < 1$, the changes would show negative autocorrelation. My hypothesis is closer to the argument

$$a_n = 1 - \frac{(X_n - X_0 - \beta_n)}{X_n}\epsilon, \quad (\epsilon > 0),$$

so that when $X_n > X_0 + \beta_n$, $a_n < 1$ a rise is more likely to be followed by a fall and when $X_n < X_0 + \beta_n$ a rise is more likely to be followed by a rise. If this were strictly true, a conventional attempt to estimate $a_n$ as a constant should be expected to result in an estimate equal to 1, and indicate zero autocorrelation despite substantial non-randomness.

My hypothesis differs from this one because I do not assume $Y_n$ to be a normal variable for all n. In my formulation $Y_n$ would be normally distributed for values of $X_n$ close to $X_0 + \beta_n$ but would be skew for much larger or smaller values of $X_n$. It is this skewness which accounts for the predicted (and observed) negative correlation.

rigid. Some professionals will be willing to buy before others, but as the price falls, the cumulative percentage of professionals willing to sell will also rise. We can investigate the nature of this phenomenon by constructing a transition matrix of probabilities of rise or decline. That is, once we detect a trend by use of the distribution of the range, we can construct a frequency distribution of the price changes conditional upon the price being a certain distance from the mean of the trend. For example, we can discover the number of times the price was five weekly standard deviations from the mean of the trend and the frequency distribution of its price change during the following week.

## THE STATISTICAL RESULTS

The statistical results I will present here are based on a sample of 45 stocks all drawn from the New York Stock Exchange. Of these 45 stocks, 26 were selected by students from a list of 50 major companies with stock option plans which was drawn up for another purpose; 13 other stocks were drawn from the steel industry[10] so as to cover as wide a range of sizes of companies as possible; 3 stocks were chosen because a stock market advisory service had referred to them as offering evidence of long trends in prices; 3 others were chosen because they exhibited strong seasonality of sales and earnings. Five of the 45 series covered a ten-year period; 40 were weekly observations for 5 years; except for one series, all of the 40 five-year stocks covered the 1956-1960 period.

All of the series were corrected for dividends by adding back all dividends, both cash and stock. As a result, the means of the price changes all include the average dividend yield on a weekly basis. Because earlier investigation had indicated that stock price changes were distributed more in accord with the log-normal distribution than the normal, all prices were converted to natural logarithms for the following computations. After all these computations had been completed it was realized that the combination of correcting for dividends and then taking logarithms tends to bias downwards the variability of observations toward the end of the period of observation. It seems unlikely that this effect was large enough to be of importance but I cannot be sure. Computations are now under way to verify the results under different methods of computation. In addition, further work is about to go forward on a random sample of seventy companies with data covering ten years.

All the tests of the autocorrelation of weekly stock price changes published so far have consistently shown deviations from random behavior, though these deviations have been uniformly small, and there is some difference in the behavior of British stock price indexes and data

[10] One additional steel company, U. S. Steel, was part of the 26-company sample drawn from the stock option group.

based on prices of individual American stocks. Kendall's data for twenty-nine British stock indexes,[11] Arnold More's data for thirty-three American companies[12] and my own data for forty-five other American companies all indicate autocorrelations which are generally small in magnitude. In each case, however, some are significantly different from zero, and what is more important is that all tend to have the same sign. In the American data for individual companies, the one- and two-week price changes show negative autocorrelation much more frequently than would be expected from a population which was truly non-autocorrelated. For the British indexes, the first three weekly price changes exhibit the same pattern except that the overwhelming proportion are positive. For slightly longer differencing intervals, Kendall's series revert to the American pattern, but it is not easy to say whether the differences are due to the fact that the British data are indexes or whether they are due to such institutional differences between the markets as the British institution of settling transactions only every two weeks.

To test this tendency toward excessive reversals, I applied the mean-square successive-differences test, which is very sensitive to this kind of non-randomness. Basically, the test is a comparison of the variance of the difference between successive one-week price changes and the variance of the price changes themselves.[13] Fourteen of the 45 series showed a significant tendency (at the 5% level) toward excessive reversals in the one-week price changes and 11 showed the same tendency in two-week price changes. Only one in the 45 price series showed a significant tendency toward positive autocorrelation at that differencing interval (Table II).

At the fourteen-week interval, the situation was almost reversed. Nine of the 45 series now showed a significant tendency for price changes to follow one another (at the 5% level) and thirty-five of the 45 series showed at least some tendency toward positive autocorrelation. The odds are more than 100 to one against such a preponderance of trends occurring if there were no such tendency in the population. Furthermore, the shift from an excessive tendency for reversals to an excessive tendency for trends takes place relatively uniformly as the interval increases.

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[11] M. Kendall.

[12] A. Moore, unpublished Ph.D. thesis, University of Chicago.

[13] The statistic used is $1 - \frac{E}{2}$, where $E = \frac{\sum_{i=1}^{n-1} (X_{i+1} - X_i)^2}{\sigma_X^2}$

and where the $X_i$ are the price changes. If the changes are independent, E should equal 2. The statistic is distributed normally with mean zero and standard deviation $\frac{n-2}{(n-1)(n+1)}$. See C. A. Bennett and N. L. Franklin, Ref. (3).

## TABLE II

## MEAN-SQUARE SUCCESSIVE DIFFERENCE TEST

$$1 - \frac{E}{2}$$

| | One-week changes | Fourteen-week changes |
|---|---|---|
| 1 | −0.276* | −0.171 |
| 2 | −0.027 | 0.116 |
| 3 | 0.019 | 0.070 |
| 4 | 0.007 | 0.214 |
| 5 | 0.005 | 0.155 |
| 6 | 0.091*** | 0.172 |
| 7 | 0.168* | 0.256 |
| 8 | −0.086*** | −0.236 |
| 9 | −0.108** | 0.068 |
| 10 | 0.015 | 0.071 |
| 11 | 0.033 | 0.390** |
| 12 | −0.170* | −0.613* |
| 13 | −0.086*** | 0.065 |
| 14 | −0.026 | −0.179 |
| 15 | 0.051 | 0.384** |
| 16 | 0.021 | −0.243 |
| 17 | −0.080*** | 0.051 |
| 18 | −0.071 | −0.010 |
| 19 | 0.043 | 0.403** |
| 20 | −0.040 | −0.316 |
| 21 | 0.038 | 0.373** |
| 22 | −0.077 | 0.289 |
| 23 | −0.333 | 0.180 |
| 24 | 0.032 | 0.490* |
| 25 | −0.084 | 0.220 |
| 26 | −0.233 | 0.496* |
| 27 | −0.084 | 0.346** |
| 28 | 0.013 | 0.025 |
| 29 | −0.162* | 0.328 |
| 30 | 0.010 | 0.068 |
| 31 | −0.337* | 0.236 |
| 32 | 0.039 | 0.187 |
| 33 | 0.037 | 0.116 |
| 34 | 0.038 | 0.428** |
| 35 | 0.011 | 0.177 |
| 36 | 0.076 | 0.254 |
| 37 | 0.061 | 0.145 |
| 38 | −0.112** | 0.215 |
| 39 | −0.092 | −0.035 |
| 40 | −0.125* | −0.410** |
| 41 | 0.076 | 0.440** |
| 42 | −0.131* | 0.325*** |
| 43 | 0.011 | 0.111 |
| 44 | −0.137* | 0.176 |
| 45 | −0.154* | 0.072 |

* Significant at 1% level
** Significant at 5% level
*** Significant at 10% level

## TABLE III

## KURTOSIS

| | | One-week changes | Fourteen-week changes |
|---|---|---|---|
| 1 | | 11.30 | 2.99 |
| 2 | | 3.19 | 1.59** |
| 3 | | 3.15 | 1.56** |
| 4 | | 4.43 | 1.81** |
| 5 | | 3.20 | 1.96** |
| 6 | | 3.39 | 1.61** |
| 7 | | 3.04 | 1.99** |
| 8 | | 6.45 | 1.81** |
| 9 | | 3.63 | 3.50 |
| 10 | | 3.08 | 2.07* |
| 11 | | 3.32 | 1.90** |
| 12 | | 3.46 | 2.04* |
| 13 | | 3.73 | 3.34 |
| 14 | | 3.10 | 2.18* |
| 15 | | 3.27 | 1.99** |
| 16 | | 3.55 | 1.78** |
| 17 | | 4.54 | 1.86** |
| 18 | | 4.43 | 1.56** |
| 19 | | 3.28 | 1.96** |
| 20 | | 3.43 | 2.48 |
| 21 | | 3.42 | 2.22* |
| 22 | | 3.73 | 2.76 |
| 23 | | 21.34 | 2.34* |
| 24 | | 4.13 | 2.41 |
| 25 | | 3.11 | 3.41 |
| 26 | | 5.65 | 2.21* |
| 27 | Steel companies | 3.17 | 2.10* |
| 28 | | 2.95 | 2.59 |
| 29 | | 15.01 | 1.85** |
| 30 | | 3.30 | 2.28* |
| 31 | | 4.75 | 2.63 |
| 32 | | 3.09 | 2.29* |
| 33 | | 3.21 | 2.45 |
| 34 | | 3.11 | 2.70 |
| 35 | | 2.98 | 2.00* |
| 36 | | 5.89 | 1.46** |
| 37 | | 3.83 | 2.02* |
| 38 | | 5.71 | 3.70 |
| 39 | | 4.11 | 2.25* |
| 40 | | 16.11 | 2.35* |
| 41 | | 3.45 | 2.05* |
| 42 | | 10.45 | 1.80** |
| 43 | | 5.31 | 2.39 |
| 44 | | 5.30 | 3.06 |
| 45 | | 3.10 | 2.18* |

** Values under 2.00
* Values less than 2.36 but greater than 2.00

As several writers have already noted, the distribution of price changes in speculative markets tends to be significantly leptokurtic. The data of my sample bear this out. Of the 45 price series tested (Table III) only two had a kurtosis less than 3 - which is the kurtosis of a normal distribution - and those two values were 2.95 and 2.98. The average kurtosis was 4.90, with three values greater than 10 and fifteen others between 4 and 10. If the successive changes were independent, we would expect that price changes over longer intervals would more closely approach the average kurtosis of a normal distribution. In fact, however, the kurtosis decreases so rapidly that it very soon falls below that of a normal distribution. At the fourteen-week interval only fourteen of the 45 series have a kurtosis greater than 2.363 which is the expected kurtosis of the small samples I have had to use. Sixteen of the series show a kurtosis less than 2.0.[14]

A complete analysis of the data on the basis of the range of a random walk[15] has not yet been possible because of certain conceptual problems. Only two companies have a trend which is significant over the entire period of observation at the 5% level. But in most cases this is because of several long segments each of which is in itself very unlikely to have come from a random sample. For example, the price of National Dairy Products in the 300 weeks beginning in January, 1950, moved in a range which was significantly too small to be random at less than the 1%

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[14] The expected kurtosis of a sample from the normal distribution is $3\left(\frac{n-1}{n+1}\right)$, where n is the sample size. In the computations in this paper, we computed $\frac{m^4}{\sigma^4}$ instead of $\frac{m^4}{s^4}$, and as a result the expected value is $\frac{(n-1)^2}{n}$ times the indicated value. For the fourteen-week kurtosis figures, n = 17. M. Kendall, Ref. (12).

It is particularly interesting to note the way that the fourteen steel companies stand out in the sample in a negative way. Only 7 out of those 14 have a lower kurtosis than the random hypothesis would suggest. The sample of steel companies is much more cyclical in its behavior than the other stocks in the sample; and, as we would expect, if a series were composed of a large number of different "trends," the resultant kurtosis would be more like that of a normal distribution. As I indicate below, the more sensitive "range test" for trends, suggests this is the case.

[15] Feller, op. cit. Feller finds two asymptotic distributions for the range: one which is appropriate if the mean of the random walk is known, another if it is unknown. The latter has the smaller sampling variance, and is the one used. In that procedure, the range is computed around the trend line found by using the observed mean price change to be the true mean. The distribution is skewed and was computed especially for this purpose. A smaller distribution for the maximum deviation is derived in Doob, Ref. (7).

The distributions derived by Feller depend upon the population standard deviation. I have derived the related distributions which use the sample standard deviation in the statistic and which are independent of the population statistics. A paper deriving, and computing these distributions will be published separately.

## TABLE IV

## AVERAGE KURTOSIS

| Number of weeks over which differences are taken | Expected value | Average (including No. 35) |
|---|---|---|
| 1 | 2.96 | 4.90 |
| 2 | 2.91 | 3.63 |
| 3 | 2.86 | 3.15 |
| 4 | 2.81 | 3.47 |
| 5 | 2.77 | 3.04 |
| 6 | 2.73 | 2.56 |
| 7 | 2.69 | 2.60 |
| 8 | 2.65 | 2.30 |
| 9 | 2.61 | 2.46 |
| 10 | 2.57 | 2.53 |
| 11 | 2.52 | 2.40 |
| 12 | 2.47 | 2.36 |
| 13 | 2.43 | 2.16 |
| 14 | 2.36 | 2.26 |

This calculation includes all stocks for a five-year period. For all companies except No. 35 (Continental Oil) the 5-year period is 1956-60. For Conoco, it is 1951-55.

level. Furthermore, from that time to the end of 1960 (209 weeks) it moved within a range around another trend which was similarly too small to be random. International Telephone and Telegraph has successive consecutive intervals (i.e., each begins where the prior one left off) of 100, 50, 125, 100, and 60, all of which are significant at the 1% level (only the first sixty weeks out of a 530-week period of observation fail to show any such significant trends.) The entire I. T. and T. series would be significantly non-random at the 12% level. General Foods shows two consecutive trends, each of which covers half of a five-year period of observation. National Biscuit, Continental Oil, Procter and Gamble, and Coca Cola all behave similarly. That is, each of these companies has only one or two trends covering all of the periods studied. Each of the forty-five companies studied has at least one trend of more than a year in length and most have at least one longer trend and very few long stretches without a trend. The more stable investment grade companies tend to have only two or three consecutive, non-overlapping trends covering all of a five- or ten-year period. The more "cyclical" companies like the steels can similarly be broken down into consecutive, non-overlapping trends, but these "trends" tend to be shorter and are sometimes separated by short intervals which do not fit into any "trend."

The conceptual problem is one of significance. Almost any series of any length has some segment which is unusual in some way. If the segments are not chosen at random, the significance of finding them is not at all clear. On the other hand, the longer (and fewer) the segments, the more significant each must be and this conclusion is strengthened if we require that the trends be consecutive, i.e., that the next one start where the last one ended. It would seem as if the lengths of the trends involved in the price series studied, especially in conjunction with the other evidence presented, are significant indications of non-randomness, but the analysis in this direction has much further to go.

Although my relatively limited investigations into commodity price behavior along these lines suggest a much closer approach to randomness, I should also point out that these results seem largely compatible with the results of Arnold Larsen's more extensive investigations into those markets. Larsen finds a tendency for shocks to be followed by reversals over short periods of time and then by weak trend effects over longer periods. His path of research, using somewhat different methods developed in collaboration with Holbrook Working, shows great promise for research in this area.

It should also be noted that evidence of trends presented here is quite compatible with a similar kind of evidence presented by Sidney Alexander and Hendrick Houthakker. These researchers found little evidence of useful short-run autocorrelations, but did find strategies which suggested that stock and commodity prices did move in trends.

Both Alexander and Houthakker attacked the problem this way. If it is true that stock prices are a random walk, there is no strategy which will, in fact, make money. If there is such a strategy price changes cannot be random. Houthakker's strategy (used on commodity futures) was: buy a security and hold it for a fixed period with a stop-loss order x% below the market. Alexander's related but more complicated strategy was: select a stock and watch it. If it goes up x%, buy it and hold on to it until it falls x% from a subsequent high in which case sell the stock you own and go short an equivalent amount. Stay short until it rises x% from a subsequent low. In this case the x% is conceived of as a filter for small, "unimportant," price changes.

Houthakker tested his rule against the behavior of actual commodity futures prices. Alexander tested his rule against movements in the Dow-Jones and Standard and Poor's indexes. Unfortunately, Alexander's rule is not nearly as effective when used on individual common stocks. Used with the indexes, gains in excess of simple strategies are achieved with "filters" as small as 5% even after allowing for commissions. For individual stocks, the filter has to be of the order of 25%. For a random sample of seventy-six stocks from the New York Stock Exchange in the period 1950-1959, gross gains (omitting commission charges) ran about 50% of that which could be achieved from a simple investment strategy.[16] This is, of course, an unfavorable period for testing any alternative to simple buying and holding, but these results are quite dismal, and indeed are intrinsically so. With a filter of 25%, declines must be very sharp indeed to permit profits on short sales.[17] Since a gain of 25% of the price of the stock must be sacrificed on a long position if the price indeed rises, substantial profits on short positions are essential if the filter technique is to be superior to simple investment. Clearly only periods including very substantial market declines would make this possible.

On the other hand, it is easy to improve upon Alexander's original and imaginative beginnings, if a model like mine proves to be true. Alexander's rule requires that the company's stock prices actually fall substantially before the stock can be sold. A rule based on a fall relative to some trend would permit much more rapid response to changes of direction. One such procedure involving the use of the probability of the range of fluctuation around the trend seems extremely promising. This involves buying (selling) the stock when its recent behavior has a low probability of arising from a random walk and selling (buying) it when that probability rises above a previously specified level. This particular strategy is very difficult to implement computationally, but short-cuts are being developed and may soon prove feasible. It has the advantage of being conceptually similar to the methods actually suggested by stock market "technicians" and thus is a fairer test of their hypotheses. From a practical point of view, it would have several advantages over the "filter" rule. First, it would enable a follower to sell (buy) a stock when it stopped rising (falling) along the previously defined trend, rather than waiting for a substantial reversal. Second, it would permit an investor the alternative of holding cash rather than adopting a position in either direction - as the filter rule (though not Houthakker's stop-loss) requires.

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16 Allen Shiner, "An Analysis of Price Movements in the Stock Market by the Filter Technique," unpublished Bachelor's thesis, M.I.T., 1962.

Actually, Shiner's results are not a literal reproduction of the Alexander hypothesis, since Shiner uses only weekly closing prices to establish highs and lows. The intraweek highs and lows would have to be more extreme, but there is no a priori way to measure the bias, since errors in both directions are possible. However, they are very unlikely to be important enough to affect these results.

As in Alexander's original paper, the gains referred to are geometric gains: i.e., they assume pyramiding of all profits.

17 Actually, for large filters Alexander uses "logarithmic filters" rather than percentages; i.e., the percentage difference from the high (low) is always measured on the low price. Thus, with a "25% filter," the price need only fall 36% to enable a short sale to make money. The short sale would take place when the price dropped to 80% of the previous high. If the price fell to 64% of the high, a repurchase would be effected when the price rose to 125% of the lower price.

While the rule I have suggested is difficult to implement, there are other simpler rules which also possess the properties I have described. One such simple decision rule is a modification of a rule actually suggested by some investment services. The rule is usually stated as follows: Compare the price today with an average of the price in the last 200 days. If the current price is higher than the moving average, buy the stock; if it is less than the moving average, sell short. If the current price rises above the moving average, cover short positions. If the price falls below the moving average, eliminate long positions.

Since the data in this study are weekly closing prices, I substituted a forty-week average for the suggested 200-day average, and compared the result of this strategy with the results of buying each stock on the 40th observed week,[18] and holding it to the end of the period of observation. The indicated strategy is much superior to simple buying and holding if only gross profits are considered. While this is strongly suggestive of non-randomness, it is not necessarily indicative of a non-randomness noticeable enough to lead to a remunerative strategy, since the moving average procedures lead to much more frequent trading than simple investment. In fact, after allowing for commissions, the moving average strategy is much inferior.

Most of the excessive transactions occur when the actual stock price remains in a narrow range. As a crude rule-of-thumb to reduce the number of transactions, the decision rule was modified to allow for transactions only when the moving average and the current price diverged by more than a certain percentage. Under this new strategy, the stock was to be bought only when the price rose above the moving average by more than 5% and would be sold whenever the price fell below the moving average by any amount: short sales would only be undertaken when the moving average rose above the price by more than 5% but would be covered whenever the price moves above the moving average by any amount. The results (Table V) show that the gross gain from this strategy is 17% greater than simple investment but the net gain is still smaller. More important, however, is the fact the average net weekly gain is substantially higher for several alternative strategies. This is because the 5% rule now leaves the investor free to divert his funds to other uses whenever the stock price shows no particular trend. For example, the net weekly gain from the moving average strategy is 9.5% greater than from simple investment.

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[18] This is the appropriate comparison for this strategy since, unlike the filter, this moving average rule can be implemented at any point of time by simple reference to the past 40 weeks of observed data. That is, no lag from the commencement of the application of the rule is necessary.

All comparisons indicated here are between sums of the logarithms of the price changes: i.e., we refer to geometric (or pyramiding) profits. These are necessarily smaller than the corresponding arithmetic sums.

## TABLE V

## RESULTS OF MOVING AVERAGE

DECISION RULE[1]
5% THRESHOLD

| | Average return per stock (%) | | Average return per stock per week (%) | | Average return per stock per week (%) (annualized) | | Average transactions per stock |
|---|---|---|---|---|---|---|---|
| | Gross | Net | Gross | Net | Gross | Net | |
| Buy and hold | 63 | 60 | 0.19 | 0.18 | 10 | 10 | 2 |
| Moving average strategy (long and short) | 79 | 54 | 0.26 | 0.20 | 14 | 11 | 15.2 |
| Moving average strategy (long)[2] | 68 | 56 | 0.38 | 0.32 | 22 | 18 | 8.0 |

ZERO THRESHOLD

| | Average return per stock (%) | | Average return per stock per week (%) | | Average return per stock per week (%) (annualized) | | Average transactions per stock |
|---|---|---|---|---|---|---|---|
| | Gross | Net | Gross | Net | Gross | Net | |
| Buy and hold | 63 | 60 | 0.19 | 0.18 | 10 | 10 | 2 |
| Moving average strategy (long and short) | 74 | 32 | 0.22 | 0.11 | 12 | 6 | 82.8 |
| Moving average strategy (long only)[1] | 68 | 48 | 0.33 | 0.25 | 19 | 14 | 40.6 |

All profits figures are geometric means: averages of the logarithms of profits reconverted to arithmetic values. Net profits are computed by assuming 1% commissions per transaction.

[1] The Decision Rule is as follows: buy the stock when the price exceeds a 40-week moving average by more than the threshold amount and sell the stock whenever the price dips below the moving average by any amount. Sell the stock short whenever it falls below the moving average by more than the threshold amount and cover the short sale whenever the price rises above the moving average by any amount.

[2] Same rule as in footnote (1), except that only long positions are included.

The weekly gain from the moving average strategy ignoring the short positions is 76% greater. In addition, the risk-averting speculator may be attracted by the fact that the variance of these strategies is some 30% less than a simple investment alternative.

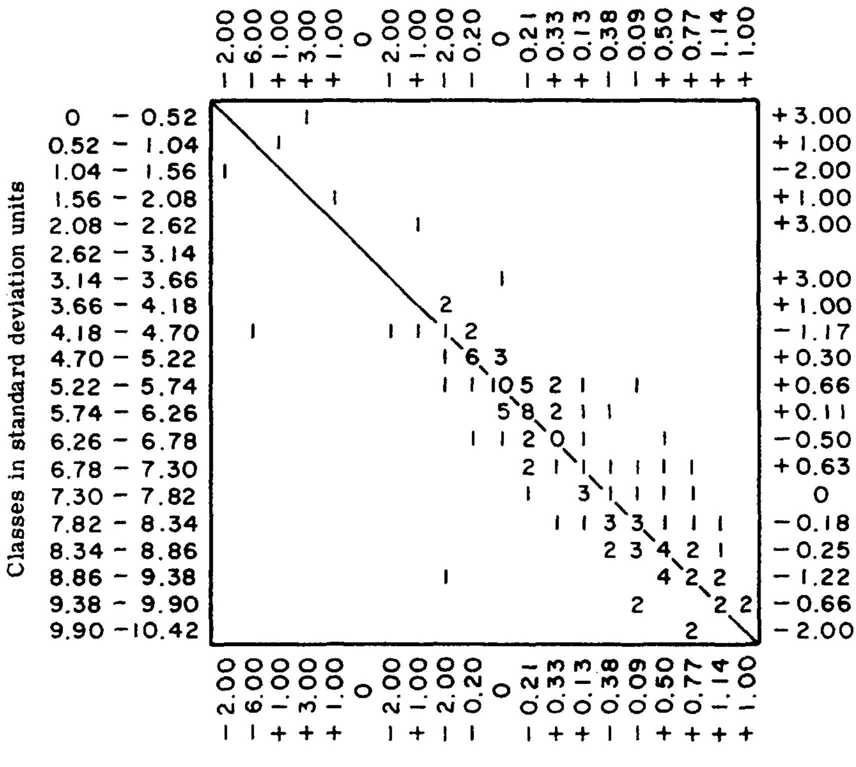
The significance of these calculations is not altogether clear. If individual stock prices were independent of one another, they would indicate very worthwhile advantages for non-random strategies. If, as is true, individual stock prices are not independent, the attractiveness of these particular strategies depends upon the alternative uses of the funds when they are not employed in common stock investment. On the other hand, these strategies are hardly exhaustive and, indeed, we have indicated some that look more promising. In addition to tests based on the range, we have not investigated moving averages of varying lengths or rules involving different thresholds. Furthermore, the period of investigation has been one which has been peculiarly satisfactory for simple investment policies,

Figure 1

IBM

Transition Matrix

Weekly, January, 1950, to March, 1952




The standard deviation is 0.0268, which means each class is about 1.4% wide.

The classes are numbered starting from the minimum deviation, so the topmost row is the lowest price class and the lowest row contains the maximum.

The outlying points are the aftermath of the Korean War outbreak.

and the sample of stocks chosen is one in which the markets are much more likely to be perfect than would be the case if smaller companies were involved.

Finally, I turn to a study of the transition matrices of price changes within the trends.

For some of the periods marked off by the trends I have detected, I have computed transition matrices for the price changes. Some of these are indicated in Figures 1 through 4. The N rows of the matrix represent N classes of prices, each of which is equal in width to $\frac{1}{N}$th of the range of the series around a trend line computed from a least-squares time trend. The minimum of the deviations from the trend is in the uppermost row; the maximum is in the lowest row. The columns are identically defined, with the lowest price class on the left and the highest on the right. The transitions are from the class denoted by row to the one denoted by the column. The diagonal entries represent movement within the same class. The entries to the right of the diagonal represent price rises. The entries to the left, price declines.

Figure 2

IBM
Transition Matrix
Weekly, March, 1952, to December, 1960

| Classes of deviations | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 2 | 4 | | | | | | | | | | | | | | | + 1.11 |
| 2 | 3 | 2 | 1 | | | | | | | | | | | | | | | − 0.33 |
| 3 | 3 | 2 | 8 | 3 | 2 | 2 | | | | | | | | | | | | + 0.25 |
| 4 | | | 6 | 13 | 8 | | | | | | | | | | | | | + 0.08 |
| 5 | | | 1 | 8 | 18 | 10 | 1 | | | | | | | | | | | + 0.05 |
| 6 | | | | 2 | 10 | 20 | 4 | 2 | 3 | | | | | | | | | + 0.07 |
| 7 | | | | 1 | | 8 | 7 | 6 | 2 | 1 | 1 | | | | | | | + 0.24 |
| 8 | | | | | | | 14 | 10 | 7 | 5 | | | | | | | | + 0.09 |
| 9 | | | | | | 1 | | 17 | 18 | 7 | 2 | 1 | | | | | | − 0.06 |
| 10 | | | | | | | | 2 | 15 | 24 | 4 | 4 | 1 | | | | | − 0.04 |
| 11 | | | | | | | | | 1 | 11 | 19 | 5 | 5 | 1 | | | | + 0.04 |
| 12 | | | | | | | | | | 2 | 12 | 10 | 3 | 2 | 2 | | | − 0.10 |
| 13 | | | | | | | | | | | 4 | 7 | 14 | 4 | | 1 | | − 0.10 |
| 14 | | | | | | | | | | | | 3 | 8 | 9 | | 1 | 1 | − 0.40 |
| 15 | | | | | | | | | | | | | | 5 | 4 | 1 | | − 0.50 |
| 16 | | | | | | | | | | | | | | 1 | 3 | 1 | 1 | − 0.66 |
| 17 | | | | | | | | | | | | | | | 1 | 2 | 5 | − 0.50 |
| | − 0.66 | 0 | − 0.10 | − 0.40 | + 0.05 | + 0.12 | − 0.30 | − 0.72 | 0 | + 0.10 | − 0.21 | + 0.10 | + 0.27 | + 0.19 | − 0.20 | + 0.66 | + 0.22 | |

Each class is about 1.6% wide

Figure 3

I. T. and T.
Transition Matrix
Weekly, January, 1950, to January, 1959

| Classes of deviations | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 1 | | | 1 | | | | | | | | | | | | | | | + 3.66 |
| 2 | 1 | 3 | 1 | 1 | | | | | | | | | | | | | | | | | + 0.33 |
| 3 | 1 | 2 | 9 | 3 | | | | | | | | | | | | | | | | | − 0.07 |
| 4 | | | 3 | 9 | 4 | 1 | | 1 | | | | | | | | | | | | | + 0.30 |
| 5 | | | 1 | 5 | 7 | 4 | 2 | | | | | | | | | | | | | | + 0.05 |
| 6 | | | | | 5 | 16 | 4 | 1 | | | | 1 | | | | | | | | | + 0.15 |
| 7 | | | | | 2 | 2 | 8 | 3 | 2 | 1 | | | | | | | | | | | + 0.33 |
| 8 | | | | | | 2 | 3 | 6 | 2 | 1 | 1 | | | | | | | | | | 0 |
| 9 | | | | | | 1 | 1 | 2 | 4 | 5 | 4 | | | | | | | | | | + 0.34 |
| 10 | | | | | | | | 2 | 7 | 15 | 8 | 2 | | | | | | | | | + 0.03 |
| 11 | | | | | | | | | 2 | 9 | 11 | 8 | 2 | | 1 | 1 | | | | | + 0.23 |
| 12 | | | | | | | | | | 3 | 10 | 31 | 18 | 3 | 1 | | | | | | + 0.06 |
| 13 | | | | | | | | | | | | 18 | 25 | 13 | 5 | 1 | | | | | + 0.11 |
| 14 | | | | | | | | | | | | 4 | 14 | 29 | 9 | 4 | 1 | | | | − 0.04 |
| 15 | | | | | | | | | | | | 2 | 3 | 12 | 12 | 4 | | 1 | | | − 0.50 |
| 16 | | | | | | | | | | | | 1 | | 3 | 6 | 6 | 1 | 3 | 1 | | − 0.30 |
| 17 | | | | | | | | | | | | | | 1 | | 4 | 5 | | | | − 0.70 |
| 18 | | | | | | | | | | | | | | | | 1 | 3 | 2 | | | − 0.80 |
| 19 | | | | | | | | | | | | | | | | | | 1 | | 1 | 0 |
| 20 | | | | | | | | | | | | | | | | | | | 1 | 4 | − 0.20 |
| | −1.50 | −0.16 | −0.14 | 0 | −0.65 | +0.20 | +0.15 | +0.20 | −0.30 | −0.16 | +0.26 | −0.27 | +0.04 | +0.03 | +0.65 | +0.66 | +0.10 | +1.14 | +1.00 | +0.20 | |

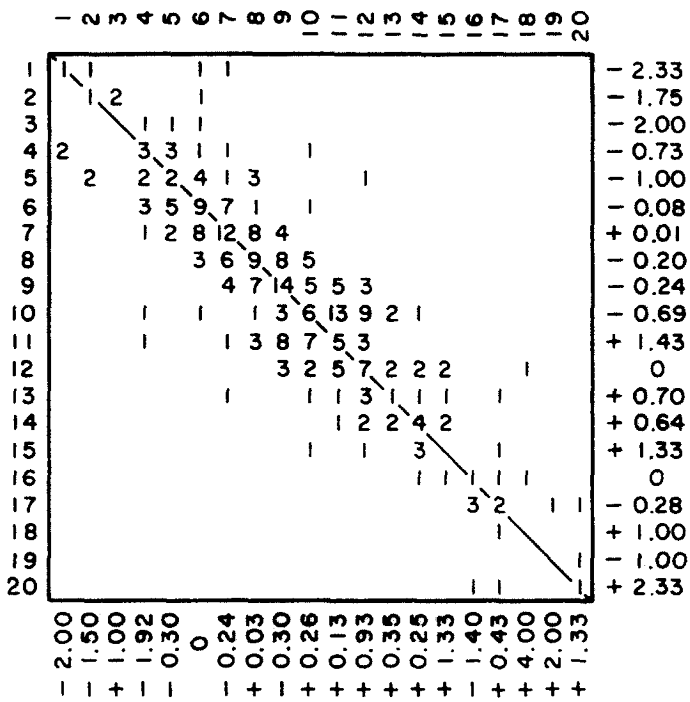
S. D. = 0.0288
Each class is about 2.3% wide

The matrices shown are typical of all those computed. There is apparently a mild tendency, especially near the maximum and minimum, for price changes to move toward the mean. This is shown by the right-hand marginal totals. There is no clear similar tendency as far as the direction from which a class is entered. The lower marginal totals indicate the average move into a given class. If the random walk hypothesis were correct, there would be the same (zero) expectation of price change in each row and each column.

The Houthakker-Alexander approaches, as well as my own tests based on the range of moving averages, all suffer from lack of a good statistical test of significance; on the other hand, they come closer to testing for the kind of non-randomness which stock market traders claim exists. It is a foolish sort of statistical reasoning which would suggest we limit our investigations to those hypotheses which are easy to investigate.

Figure 4

NATIONAL DAIRY PRODUCTS
Transition Matrix
Weekly, January, 1950, to October, 1955




Each class is about 1.4% wide

The way in which actual markets operate is one of the more fascinating of current economic questions. If their behavior is more complicated than the random walk models suggest, it will take more sophisticated statistical testing to discover it.

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