THICK POINTS FOR PLANAR BROWNIAN MOTION AND THE ERDŐS-TAYLOR CONJECTURE ON RANDOM WALK

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ABSTRACT. Let $\mathcal{T}(x,r)$ denote the occupation measure of the disc of radius r centered at x by planar Brownian motion run till time 1. We prove that $\sup_{|x|\le 1}\mathcal{T}(x,r)/(r^2|\log r|^2)\to 2$ a.s. as $r\to 0$, thus solving a problem posed by Perkins and Taylor (1987). Furthermore, we show that for any $a<2$, the Hausdorff dimension of the set of "perfectly thick points" x for which $\lim_{r\to 0}\mathcal{T}(x,r)/(r^2|\log r|^2)=a$, is almost surely $2-a$; this is the correct scaling to obtain a nondegenerate "multifractal spectrum" for Brownian occupation measure in the plane. The proofs rely on a 'multiscale refinement' of the second moment method. As a consequence of our results on Brownian motion, we prove a conjecture about simple random walk in $\mathbf{Z}^2$ due to Erdős and Taylor (1960): The number of visits to the most frequently visited lattice site in the first n steps of the walk, is asymptotic to $(\log n)^2/\pi$. We also determine the corresponding "discrete multifractal spectrum": For $0<\alpha<1/\pi$, the number of points visited more than $\alpha(\log n)^2$ times in the first n steps of the walk, is $n^{1-\alpha\pi+o(1)}$.

1. Introduction

Forty years ago, Erdős and Taylor (1960) posed a problem about simple random walks in $\mathbf{Z}^2$: *How many times does the walk revisit the most frequently visited site in the first n steps?* Denote by $T_n(x)$ the number of visits of planar simple random walk to x by time n, and let $T_n^* := \max_{x\in\mathbf{Z}^2} T_n(x)$. Erdős and Taylor [7, (3.11)] proved that

$$\frac{1}{4\pi}\le \liminf_{n\to\infty}\frac{T_n^*}{(\log n)^2}\le \limsup_{n\to\infty}\frac{T_n^*}{(\log n)^2}\le\frac{1}{\pi}\quad a.s., \tag{1.1}$$

and conjectured that the limit exists and equals $1/\pi$ a.s. The importance of determining the value of this limit is clarified in (1.3) below, where this value appears in the power laws governing the local time of the walk.

The Erdős-Taylor conjecture was quoted in the book by Révész [19, Section 19.2] but to the best of our knowledge, the bounds in (1.1) were not improved prior to the present paper. As it turns out, an important step

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towards our solution of the Erdős-Taylor conjecture was the formulation by Perkins and Taylor (1987) of an analogous problem on the maximal occupation measure that planar Brownian motion (run for unit time) can assign to discs of a given radius. Perkins and Taylor also obtained upper and lower bounds with a ratio of 4 between them, and conjectured in [17, Conjecture 2.4] that the upper bound is sharp.

In this paper we prove this conjecture of Perkins and Taylor as part of a study of the fine multifractal structure of Brownian occupation measure. The proof is based on a 'multiscale refinement' of the classical second moment method; since the second moment method is such a widely used tool in probability, we believe that our refinement will have further applications to other problems where the standard second moment method breaks down due to high correlations.

We then establish the Erdős-Taylor conjecture by using strong approximation. Indeed, this derivation highlights the significance of the Komlós-Major-Tusnády [11] strong approximation theorems, and their multidimensional extensions by Einmahl [6]; earlier approximations are not sharp enough to obtain the Erdős-Taylor conjecture from our Brownian motion results.

Although the bulk of our work is in the Brownian motion setting, we first state our results for simple random walk. A generalization to a class of planar random walks is stated and proven in Section 5.

Theorem 1.1. *Let $S_n = \sum_{i=1}^n X_i$ denote simple random walk in $\mathbf{Z}^2$. Let $M(n,\alpha)$ denote the number of points in the set $\{x : T_n(x) \geq \alpha(\log n)^2\}$. Then,*

$$\lim_{n\to\infty} \frac{T_n^*}{(\log n)^2} = \frac{1}{\pi}, \quad a.s., \tag{1.2}$$

and for $\alpha \in (0, 1/\pi]$,

$$\lim_{n\to\infty} \frac{\log M(n,\alpha)}{\log n} = 1 - \alpha\pi \quad a.s. \tag{1.3}$$

Moreover, any (random) sequence $\{x_n\}$ in $\mathbf{Z}^2$ such that $T_n(x_n)/T_n^* \to 1$, must satisfy

$$\lim_{n\to\infty} \frac{\log |x_n|}{\log n} = \frac{1}{2} \quad a.s. \tag{1.4}$$

The last assertion of the theorem improves an estimate of Révész [19, Theorem 22.8], and shows that the "favourite points" for planar simple random walk by time n, are *consistently* located near the boundary of the range (on a logarithmic scale); the analogous statement for simple random walk on $\mathbf{Z}$ is contained in a well-known result of Bass and Griffin [1].

Next, we collect some definitions needed to state our Brownian motion results. For any Borel measurable function f from $0 \leq t \leq T$ to $\mathbb{R}^2$, denote by μ^f_T its *occupation measure*:

$$\mu^f_T(A) = \int_0^T \mathbf{1}_A(f_t)\,dt$$

for all Borel sets $A \subseteq \mathbb{R}^2$. Throughout, $D(x,r)$ denotes the open disc in $\mathbb{R}^2$ of radius r centered at x, and $\{w_t\}_{t\geq 0}$ denotes planar Brownian motion started at the origin. Let $\bar\theta = \inf\{t : |w_t| = 1\}$. We write $\dim(A)$ for the Hausdorff dimension of a set A.

Our results for planar Brownian motion follow; the first one was conjectured by Perkins and Taylor [17].

Theorem 1.2.

$$\lim_{\varepsilon\to 0}\sup_{x\in\mathbb{R}^2} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = 2\,, \qquad a.s. \tag{1.5}$$

Here one may replace $\bar\theta$ by any deterministic $0 < T < \infty$; this is the form in which this problem was stated as [17, Conjecture 2.4]. This theorem should be compared with the classical result of Ray, [18, Theorem 1]:

$$\limsup_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(0,\varepsilon))}{\varepsilon^2\log\frac{1}{\varepsilon}\log\log\log\frac{1}{\varepsilon}} = \frac{1}{2} \qquad a.s. \tag{1.6}$$

Theorem 1.2 has an application to the problem of reconstructing the range of spatial Brownian motion from the occupation measure projected to a sphere; see Pemantle et al [16].

Recall that for almost all Brownian paths w, the pointwise Hölder exponent

$$\mathsf{H\ddot{o}lder}(\mu^w_{\bar\theta}, x) := \lim_{\epsilon\to 0} \frac{\log\mu^w_{\bar\theta}(D(x,\epsilon))}{\log\epsilon} \tag{1.7}$$

takes the value 2 for *all* points x in the range $\{w_t \,\big|\, 0 \leq t \leq \bar\theta\}$. Hence, as explained in [3], standard multifractal analysis must be refined in order to capture the delicate fluctuations of Brownian occupation measure and obtain a nondegenerate dimension spectrum. However, the logarithmic corrections required for spatial and planar Brownian motion are different. The next theorem describes the multifractal structure of planar occupation measure.

Theorem 1.3. *For any $a \leq 2$,*

$$\dim\left\{x\in D(0,1) : \lim_{\varepsilon\to 0}\frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = a\right\} = 2 - a \qquad a.s. \tag{1.8}$$

Equivalently

$$\dim\left\{0\leq t\leq\bar\theta : \lim_{\varepsilon\to 0}\frac{\mu^w_{\bar\theta}(D(w_t,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = a\right\} = 1 - a/2 \qquad a.s. \tag{1.9}$$

Also,

$$\sup_{|x|<1} \limsup_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = 2 \qquad a.s. \tag{1.10}$$

Remarks.

- We call a point $x \in D(0,1)$ on the Brownian path a *perfectly thick point* if x is in the set considered in (1.8) for some $a > 0$; similiarly, $t > 0$ is called a perfectly thick time if it is in the set considered in (1.9) for some $a > 0$.
- Perhaps of greater significance than the numerical results in the theorems above, are the insights that their proofs yield on the nature of thick points in the plane and the contrast with the spatial case. In our study [3] of thick points for spatial Brownian motion, a key role was played by a certain localization phenomenon: The balls of radius ϵ that have the largest occupation measure accumulate most of this measure in a short time interval (of length at most $\epsilon^2|\log\epsilon|^b$ for some b); This localization does not hold in the planar case, where the balls of radius ϵ with greatest occupation measure accumulate this measure on a macroscopic time interval (of length longer than ϵ^γ for any $\gamma > 0$). During this time interval, the Brownian particle makes excursions of essentially all length scales ϵ^γ. These excursions create substantial dependence between occupation measures of rather distant discs; handling this dependence is the crux of our work.
- By Brownian scaling, for any determinsitic $0 < r < \infty$, the set $D(0,1)$ and $\bar\theta$ can be replaced by $D(0,r)$ and $\bar\theta_r = \inf\{t : |w_t| = r\}$, without changing the conclusion of Theorem 1.3. Similarly, one may replace $\mu^w_{\bar\theta}$ by μ^w_T in the statement of the theorem, for any deterministic $T < \infty$.
- For any $x \notin \{w_t \,\big|\, 0 \le t \le \bar\theta\}$ and ϵ small enough, $\mu^w_{\bar\theta}(D(x,\epsilon)) = 0$. Hence, the equivalence of (1.8) and (1.9) is a direct consequence of the *uniform dimension doubling* property of Brownian motion, due to Kaufman [9] (see also, [17, Eqn. (0.1)]).
- The proof of our theorem will also show that

$$\dim\left\{x \in D(0,1) : \limsup_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = a\right\} = 2 - a \qquad a.s., \tag{1.11}$$

and

$$\dim\left\{x \in D(0,1) : \liminf_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} = a\right\} = 2 - a \qquad a.s. \tag{1.12}$$

This is in contrast to the situation for transient Brownian motion, [3], where the lim sup and lim inf results analogous to (1.11) and (1.12) require different scalings.

We call a point $x \in D(0,1)$ on the Brownian path a *thick point* if x is in the set considered in (1.11) for some $a > 0$, and a *consistently thick point* if x is in the set considered in (1.12) for some $a > 0$.

- Similarly we will see that

$$\dim\left\{x \in D(0,1) : \limsup_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} \geq a\right\} = 2-a \qquad a.s.\,, \tag{1.13}$$

and

$$\dim\left\{x \in D(0,1) : \liminf_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} \geq a\right\} = 2-a \qquad a.s. \tag{1.14}$$

As in the case of spatial Brownian motion, we have the following analogue of the coarse multi-fractal spectrum:

Theorem 1.4. *For all $a < 2$,*

$$\lim_{\varepsilon\to 0} \frac{\log \mathcal{L}eb(x : \mu^w_{\bar\theta}(D(x,\varepsilon)) \geq a\varepsilon^2(\log\varepsilon)^2)}{\log\varepsilon} = a\,, \quad a.s.$$

The basic approach of this paper, which goes back to Ray, [18], is to control occupation times using excursions between concentric discs. The number of excursions between discs centered at a thick point is so large, that the occupation times will necessarily be concentrated near their conditional means given the excursion counts (see Lemma 3.1). Section 2 provides a simple lemma which will be useful in exploiting this link between excursions and occupation times. This lemma is then used to obtain the upper bounds in Theorems 1.2 and 1.3. In Section 3 we explain how to obtain the analogous lower bounds, leaving technical details to lemmas which are proven in later sections. The key idea in the proof of the lower bound, is to control excursions on many scales simultaneously, leading to a 'multiscale refinement' of the classical second moment method. This is inspired by techniques from probability on trees, in particular the analysis of first-passage percolation by Lyons and Pemantle [12]. The approximate tree structure that we (implicitly) use arises by considering discs of the same radius r around different centers and varying r; for fixed centers x, y, and "most" radii r (on a logarithmic scale) the discs $D(x,r)$ and $D(y,r)$ are either well-separated (if $r << |x-y|$) or almost coincide (if $r >> |x-y|$).

In Section 4 we prove Theorem 1.4 on the coarse multifractal spectrum, while in Section 5 we prove Theorem 5.1 (and in particular, the Erdős-Taylor conjecture). Sections 6-9 establish the technical lemmas used in Section 3. Complements and open problems are collected in the final section.

2. Hitting time estimates and upper bounds

The following simple lemma will be used repeatedly. Throughout this section, fix $0 < r_1 \leq r_3$, let $\bar\sigma = \inf\{t > 0 : |w_t| = r_3\}$, and define

$$\bar\tau = \int_0^{\bar\sigma} 1_{D(0,r_1)}(w_s)\,ds\,.$$

Lemma 2.1. *For* $|x_0| = r_2$,

$$\mathbb{E}^{x_0}(\bar\tau/r_1^2) = \log(r_3/r_2)\,, \qquad \text{for} \quad r_1 \le r_2 \le r_3. \tag{2.1}$$

For all $k \ge 1$,

$$\mathbb{E}^{x_0}(\bar\tau/r_1^2)^k \le k![\log(r_3/r_1) + \frac{1}{2}]^k\,, \tag{2.2}$$

implying that for $0 \le \lambda < r_1^{-2}/[\log(r_3/r_1) + \frac{1}{2}]$,

$$\mathbb{E}^{x_0}(e^{\lambda\bar\tau}) \le (1 - \lambda r_1^2[\log(r_3/r_1) + \frac{1}{2}])^{-1}\,. \tag{2.3}$$

Proof of Lemma 2.1: By Brownian scaling, we may and shall assume without loss of generality that $r_1 = 1$. Due to radial symmetry, $E^x(\bar\tau) = u(|x|)$ is a function of $|x|$ only, with $u(r)$ satisfying

$$\left\{ \begin{array}{l} \frac{1}{2}(u'' + r^{-1}u') = -\mathbf{1}_{r\le 1} \\ \quad u(r_3) = 0\,, \end{array} \right. \tag{2.4}$$

for $r \in [0, r_3]$. Solving for (2.4), one finds that

$$u(r) = \left\{ \begin{array}{ll} \frac{-r^2}{2} + \frac{1}{2} + \log r_3\,, & r \le 1 \\ \log r_3 - \log r\,, & r_3 \ge r \ge 1\,, \end{array} \right. \tag{2.5}$$

proving (2.1). Since $u(r) \le \frac{1}{2} + \log r_3$, we have by the strong Markov property that

$$\begin{aligned} \mathbb{E}^{x_0}(\bar\tau^k) &= k!\mathbb{E}^{x_0}\left(\int_{0\le s_1\le\cdots\le s_k\le\bar\sigma} \prod_{i=1}^{k} 1_{D(0,1)}(w_{s_i})ds_1\cdots ds_k\right) \\ &= k!\mathbb{E}^{x_0}\left(\int_{0\le s_1\le\cdots\le s_{k-1}\le\bar\sigma} \prod_{i=1}^{k-1} 1_{D(0,1)}(w_{s_i})u(|w_{s_{k-1}}|)ds_1\cdots ds_{k-1}\right) \\ &\le k(\frac{1}{2} + \log r_3)\mathbb{E}^{x_0}(\bar\tau^{k-1})\,, \end{aligned}$$

proving (2.2) by induction on k. The bound (2.3) then follows by the power series expansion of $e^{\lambda\bar\tau}$. □

We next provide the required upper bounds in Theorems 1.2 and 1.3. Namely, with the notation

$$\mathsf{Thick}_{\ge a} = \left\{ x \in D(0,1) :\ \limsup_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} \ge a \right\}, \tag{2.6}$$

we will show that for any $a \in (0, 2]$,

$$\dim(\mathsf{Thick}_{\ge a}) \le 2 - a\,, \quad a.s.\,, \tag{2.7}$$

and

$$\limsup_{\varepsilon\to 0}\ \sup_{|x|<1} \frac{\mu^w_{\bar\theta}(D(x,\varepsilon))}{\varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^2} \le 2\,, \quad a.s. \tag{2.8}$$

(note that (2.8) provides the upper bound also for (1.10)).

Set $h(\epsilon)=\epsilon^2|\log\epsilon|^2$ and

$$z(x,\epsilon):=\mu^w_{\bar\theta}(D(x,\varepsilon))/h(\varepsilon).$$

Fix $\delta>0$ small enough ($\delta<1/22$ will do), and choose a sequence $\tilde\epsilon_n\downarrow 0$ as $n\to\infty$ in such a way that $\tilde\epsilon_1<e^{-1}$ and

$$h(\tilde\epsilon_{n+1})=(1-\delta)h(\tilde\epsilon_n), \tag{2.9}$$

implying that $\tilde\epsilon_n$ is monotone decreasing in n. Since, for $\tilde\epsilon_{n+1}\le\epsilon\le\tilde\epsilon_n$ we have

$$z(x,\tilde\epsilon_n)=\frac{h(\tilde\epsilon_{n+1})}{h(\tilde\epsilon_n)}\frac{\mu^w_{\bar\theta}(D(x,\tilde\epsilon_n))}{h(\tilde\epsilon_{n+1})}\ge(1-\delta)z(x,\epsilon)\,, \tag{2.10}$$

it is easy to see that for any $a>0$,

$$\mathsf{Thick}_{\ge a}\subseteq D_a:=\{x\in D(0,1)\;\Big|\;\limsup_{n\to\infty}z(x,\tilde\epsilon_n)\ge(1-\delta)a\}.$$

Let $\{x_j:j=1,\dots,\bar K_n\}$, denote a maximal collection of points in $D(0,1)$ such that $\inf_{\ell\ne j}|x_\ell-x_j|\ge\delta\tilde\epsilon_n$. Let $\bar\theta_2=\inf\{t:|w_t|=2\}$ and $\mathcal{A}_n$ be the set of $1\le j\le\bar K_n$, such that

$$\mu^w_{\bar\theta}(D(x_j,(1+\delta)\tilde\epsilon_n))\ge(1-2\delta)ah(\tilde\epsilon_n). \tag{2.11}$$

Applying (2.3) for $r_1=(1+\delta)\tilde\epsilon_n$, $r_3=2$ and $\lambda=(1+\delta)^{-1}r_1^{-2}/|\log\tilde\epsilon_n|$ it follows by Chebyscheff's inequality that

$$\mathbf{P}^x(\mu^w_{\bar\theta_2}(D(0,(1+\delta)\tilde\epsilon_n))\ge(1-2\delta)ah(\tilde\epsilon_n))\le c\,\tilde\epsilon_n^{(1-10\delta)a}\,,$$

for some $c=c(\delta)<\infty$, all sufficiently large n and any $x\in D(0,1)$. Note that for all $x\in D(0,1)$ and $\epsilon,b\ge0$

$$\mathbf{P}(\mu^w_{\bar\theta}(D(x,\epsilon))\ge b)\le\mathbf{P}^{-x}(\mu^w_{\bar\theta_2}(D(0,\epsilon))\ge b)\,.$$

Thus, for all sufficiently large n, any j and $a>0$,

$$\mathbf{P}(j\in\mathcal{A}_n)\le c\,\tilde\epsilon_n^{(1-10\delta)a}\,, \tag{2.12}$$

implying that

$$\mathbb{E}|\mathcal{A}_n|\le c'\,\tilde\epsilon_n^{(1-10\delta)a-2}. \tag{2.13}$$

Let $\mathcal{V}_{n,j}=D(x_j,\delta\tilde\epsilon_n)$. For any $x\in D(0,1)$ there exists $j\in\{1,\dots,\bar K_n\}$ such that $x\in\mathcal{V}_{n,j}$, hence $D(x,\tilde\epsilon_n)\subseteq D(x_j,(1+\delta)\tilde\epsilon_n)$. Consequently, $\cup_{n\ge m}\cup_{j\in\mathcal{A}_n}\mathcal{V}_{n,j}$ forms a cover of D_a by sets of maximal diameter $2\delta\tilde\epsilon_m$. Fix $a\in(0,2]$. Since $\mathcal{V}_{n,j}$ have diameter $2\delta\tilde\epsilon_n$, it follows from (2.12) that for $\gamma=2-(1-11\delta)a>0$,

$$\mathbb{E}\sum_{n=m}^\infty\sum_{j\in\mathcal{A}_n}|\mathcal{V}_{n,j}|^\gamma\le c'\,(2\delta)^\gamma\sum_{n=m}^\infty\tilde\epsilon_n^{\delta a}<\infty\,.$$

Thus, $\sum_{n=m}^\infty\sum_{j\in\mathcal{A}_n}|\mathcal{V}_{n,j}|^\gamma$ is finite a.s. implying that $\dim(D_a)\le\gamma$ a.s. Taking $\delta\downarrow0$ completes the proof of the upper bound (2.7).

Turning to prove (2.8), set $a = (2+\delta)/(1-10\delta)$ noting that by (2.13)

$$\sum_{n=1}^{\infty} \mathbf{P}(|\mathcal{A}_n| \geq 1) \leq \sum_{n=1}^{\infty} \mathbb{E}|\mathcal{A}_n| \leq c' \sum_{n=1}^{\infty} \tilde{\epsilon}_n^{\,\delta} < \infty .$$

By Borel-Cantelli, it follows that almost surely, $\mathcal{A}_n$ is empty for all $n > n_0(\omega)$ and some $n_0(\omega) < \infty$. By (2.10) we then have

$$\sup_{\epsilon \leq \tilde{\epsilon}_{n_0(\omega)}} \sup_{|x|<1} \frac{\mu_{\bar{\theta}}^{w}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} \leq a ,$$

and (2.8) follows by taking $\delta \downarrow 0$. □

3. Lower bounds

Fixing $a < 2$, $c > 0$ and $\delta > 0$, let $\bar{\theta}_c = \bar{\theta}_c(w) = \inf\{t : |w_t| = c\}$,

$$\Gamma_c = \Gamma_c(w) := \{x \in D(0,c) : \lim_{\varepsilon \to 0} \frac{\mu_{\bar{\theta}_c}^{w}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} = a\} ,$$

and $\mathcal{E}_c := \{w : \dim(\Gamma_c(w)) \geq 2 - a - \delta\}$.

In view of the results of Section 2, we will obtain Theorem 1.3 and (1.11)-(1.14) once we show that $\mathbf{P}(\mathcal{E}_1) = 1$ for any $a < 2$ and $\delta > 0$. Moreover, then the inequality

$$\liminf_{\epsilon \to 0} \sup_{|x|<1} \frac{\mu_{\bar{\theta}}^{w}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} \geq \sup_{|x|<1} \liminf_{\epsilon \to 0} \frac{\mu_{\bar{\theta}}^{w}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2}$$

implies that for any $\eta > 0$,

$$\liminf_{\varepsilon \to 0} \sup_{|x|<1} \frac{\mu_{\bar{\theta}}^{w}(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} \geq 2(1-\eta) , \quad a.s.$$

In view of (2.8), these lower bounds establish Theorem 1.2.

The bulk of this section will be dedicated to showing that $\mathbf{P}(\mathcal{E}_1) > 0$. Assuming this for the moment, let us show that this implies $\mathbf{P}(\mathcal{E}_1) = 1$. With $w_t^c := c^{-1} w_{c^2 t}$ we have that $c^2 \bar{\theta}(w^c) = \inf\{c^2 t : |c^{-1} w_{c^2 t}| = 1\} = \bar{\theta}_c(w)$ and hence

$$\begin{aligned}
\mu_{\bar{\theta}}^{w^c}(D(x,\varepsilon)) &= \int_0^{\bar{\theta}(w^c)} 1_{\{|w_s^c - x| \leq \epsilon\}}\, ds = \int_0^{\bar{\theta}(w^c)} 1_{\{|w_{c^2 s} - cx| \leq c\epsilon\}}\, ds \\
&= \frac{1}{c^2} \int_0^{c^2 \bar{\theta}(w^c)} 1_{\{|w_s - cx| \leq c\epsilon\}}\, ds = \frac{1}{c^2} \mu_{\bar{\theta}_c}^{w}(D(cx, c\varepsilon)).
\end{aligned}$$

Consequently, $\Gamma_c(w) = c\Gamma_1(w^c)$, so Brownian scaling implies that $p = \mathbf{P}(\mathcal{E}_c)$ is independent of $c > 0$. Let

$$\mathcal{E} := \limsup_{n \to \infty} \mathcal{E}_{n^{-1}} ,$$

so that $\mathbf{P}(\mathcal{E}) \geq p$. Since $\mathcal{E}_c \in \mathcal{F}_{\bar{\theta}_c}$ and $\bar{\theta}_{n^{-1}} \downarrow 0$, the Blumenthal $0-1$ law tells us that $\mathbf{P}(\mathcal{E}) \in \{0,1\}$. Thus, $p > 0$ yields $\mathbf{P}(\mathcal{E}) = 1$. We will see momentarily that the events $\mathcal{E}_c$ are essentially increasing in c, *i.e.*,

$$\forall 0 < b < c \quad \mathbf{P}(\mathcal{E}_b \setminus \mathcal{E}_c) = 0\,. \tag{3.1}$$

Thus, $\mathbf{P}(\mathcal{E} \setminus \mathcal{E}_1) \leq \mathbf{P}(\bigcup_n \{\mathcal{E}_{n^{-1}} \setminus \mathcal{E}_1\}) = 0$, so that also $\mathbf{P}(\mathcal{E}_1) = 1$.

To see (3.1), observe that for $b < c$,

$$\Gamma_b(w) \setminus \{w_t : \bar{\theta}_b \leq t \leq \bar{\theta}_c\} \subset \Gamma_c(w)\,.$$

Hence, with $\mathcal{F}_{\bar{\theta}_b} = \sigma(\{w_t : 0 \leq t \leq \bar{\theta}_b\})$,

$$\mathbf{P}(\mathcal{E}_b \setminus \mathcal{E}_c) \leq \mathbb{E}\mathbf{P}(\dim(\Gamma_b(w)) \neq \dim(\Gamma_b(w) \setminus \{w_t : \bar{\theta}_b \leq t \leq \bar{\theta}_c\})\,|\mathcal{F}_{\bar{\theta}_b})\,.$$

Applying the strong Markov property at time $\bar{\theta}_b$ and observing that the set $\Gamma_b(w)$ is a.s. analytic, we thus obtain (3.1) as a consequence of a general fact:

Any fixed planar analytic set A satisfies

$$\dim(A \setminus [w]) = \dim(A) \quad a.s. \tag{3.2}$$

where $[w] := \{w_t : t > 0\}$ is the range of planar Brownian motion w started at any fixed point.

To verify (3.2), suppose that $\dim(A) > \alpha$. Then there exists a Frostman measure ν on A, *i.e.*, a positive finite measure such that $\nu(B) \leq (\mathrm{diam} B)^\alpha$ for all balls B; see [8, page 130]. Since w does not hit points, Fubini's theorem yields that

$$\mathbb{E}(\nu([w])) = \mathbb{E}\int \mathbf{1}_{\{x \in [w]\}}\, d\nu(x) = \int \mathbf{P}(x \in [w])\, d\nu(x) = 0\,.$$

Thus ν is a.s. carried by $(A \setminus [w])$, whence $\dim(A \setminus [w]) \geq \alpha$ a.s. This proves (3.2).

It thus only remains to show that $\mathbf{P}(\mathcal{E}_1) > 0$. We start by constructing a subset of Γ_1, the Hausdorff dimension of which is easier to bound below. To this end, fix $\epsilon_1 = 1/8$ and the square $S = [\epsilon_1, 2\epsilon_1]^2 \subset D(0,1)$. Note that for all $x \in S$ and $y \in S \cup \{0\}$ both $0 \notin D(x,\epsilon_1)$ and $0 \in D(x,1/2) \subset D(y,1) \subset D(x,2)$. Let $\epsilon_k = \epsilon_1 (k!)^{-3} = \epsilon_1 \prod_{l=2}^k l^{-3}$. For $x \in S$, $k \geq 2$ and $\rho > \epsilon_1$ let $N_k^x(\rho)$ denote the number of excursions from $\partial D(x,\epsilon_{k-1})$ to $\partial D(x,\epsilon_k)$ prior to hitting $\partial D(x,\rho)$. Set $n_k = 3ak^2 \log k$. We will say that a point $x \in S$ is **n-perfect** if

$$n_k - k \leq N_k^x(1/2) \leq N_k^x(2) \leq n_k + k\,, \qquad \forall k = 2,\dots,n. \tag{3.3}$$

For $n \geq 2$ we partition S into $M_n = \epsilon_1^2/(2\epsilon_n)^2 = (1/4)\prod_{l=1}^n l^6$ non-overlapping squares of edge length $2\epsilon_n = 2\epsilon_1/(n!)^3$, which we denote by $S(n,i)\,;\, i = 1,\dots,M_n$ with $x_{n,i}$ denoting the center of each $S(n,i)$. Let $Y(n,i)\,;\, i = 1,\dots,M_n$ be the sequence of random variables defined by

$$Y(n,i) = 1 \ \text{ if } x_{n,i} \text{ is n-perfect}$$

and $Y(n,i)=0$ otherwise. Set $q_{n,i}=P(Y(n,i)=1)=E(Y(n,i))$. Define

$$A_n = \bigcup_{i:Y(n,i)=1} S(n,i), \tag{3.4}$$

$$F_m = \overline{\bigcup_{n\geq m} A_n}\,, \tag{3.5}$$

and

$$F = F(\omega) = \bigcap_m F_m. \tag{3.6}$$

Note that each $x\in F$ is the limit of a sequence $\{x_n\}$ such that x_n is n-perfect. Since

$$D(x_n,\epsilon-|x-x_n|)\subset D(x,\epsilon)\subset D(x_n,\epsilon+|x-x_n|)$$

for $x\in F$, applying the next lemma (to be proven in Section 6), for the n-perfect points x_n and using the continuity of $\epsilon\mapsto\epsilon^2|\log\epsilon|^2$ we conclude that $F\subset\Gamma_1$.

Lemma 3.1. *There exists a $\delta(\epsilon)=\delta(\epsilon,\omega)\to 0$ a.s. such that for all m and all $x\in S$, if x is m-perfect then*

$$a-\delta(\epsilon)\leq\frac{\mu_{\bar\theta}^{w}(D(x,\epsilon))}{\epsilon^2(\log\epsilon)^2}\leq a+\delta(\epsilon), \qquad \forall\epsilon\geq\epsilon_m. \tag{3.7}$$

To complete the proof that $\mathbf{P}(\mathcal{E}_1)>0$ it thus suffices to show that

$$\mathbf{P}(\dim(F)\geq 2-a-\delta)>0, \tag{3.8}$$

for any $a<2$ and $\delta>0$. Fixing $a<2$ and $\delta>0$ such that $h:=2-a-\delta>0$, we establish (3.8) by finding a set $\mathcal{C}$ of positive probability, such that for any $\omega\in\mathcal{C}$ we can find a non-zero random measure ρ_ω supported on $F(\omega)$ with finite h-energy, where the h-energy of a measure ν is defined as

$$\mathcal{G}_h(\nu)=\int\int|x-y|^{-h}\,d\nu(x)\,d\nu(y) \tag{3.9}$$

(see e.g. [13, Theorem 8.7]). The measure $\rho=\rho_\omega$ shall be constructed as a weak limit of measures ν_n, where $\nu_n=\nu_{n,\omega}$ for $n\geq 2$ is the random measure supported on $A_n\subseteq F_n$ whose density with respect to Lebesgue measure is

$$f_n(x)=\sum_{i=1}^{M_n} q_{n,i}^{-1}1_{\{Y(n,i)=1\}}1_{\{x\in S(n,i)\}}.$$

Note that

$$\mathbb{E}\left(\nu_n(S)\right)=\sum_{i=1}^{M_n} q_{n,i}^{-1}P(Y(n,i)=1)(2\epsilon_n)^2=\epsilon_1^2. \tag{3.10}$$

Observe that if $x\in S$ is n-perfect then the number N_k^x of excursions from $\partial D(x,\epsilon_{k-1})$ to $\partial D(x,\epsilon_k)$ *prior to* $\bar\theta$ is also between n_k-k and n_k+k. Whereas it is this property that leads to Lemma 3.1, the use of a stopping time related to the x-concentric disks in the definition of $N_k^x(\rho)$ simplifies the task of estimating first and second moments of $Y(n,i)$. These estimates, summarized in the next lemma, are a direct consequence of Lemmas 7.1 and 8.1.

Lemma 3.2. *Let $l(i,j) = \min\{m : D(x_{n,i},\epsilon_m) \cap D(x_{n,j},\epsilon_m) = \emptyset\} \leq n$. There exists $\delta_n \to 0$ such that for all $n \geq 2$, i,*

$$q_{n,i} \geq Q_n := \inf_{x\in S} \mathbf{P}(x \text{ is } n\text{-perfect}) \geq \epsilon_n^{a+\delta_n}\ , \tag{3.11}$$

whereas for all n and $i \neq j$,

$$\mathbb{E}(Y(n,i)Y(n,j)) \leq Q_n^2 \epsilon_{l(i,j)}^{-a-\delta_{l(i,j)}}\ . \tag{3.12}$$

Furthermore, $Q_n \geq cq_{n,i}$ for some $c > 0$ and all $n \geq 2$ and i.

In the sequel, we let C_m denote generic finite constants that are independent of n. The definition of $l(i,j) \geq 2$ implies that

$$2\epsilon_{l(i,j)} \leq |x_{n,i} - x_{n,j}| \leq 2\epsilon_{l(i,j)-1}. \tag{3.13}$$

Recall that there are at most $C_0\epsilon_{l-1}^2\epsilon_n^{-2} = C_0 l^6 \epsilon_l^2 \epsilon_n^{-2}$ points $x_{n,j}$ in the ball of radius $2\epsilon_{l-1}$ centered at $x_{n,i}$. Taking hereafter $l(i,i) := n$, the last statement of Lemma 3.2 shows that (3.12) holds (up to a multiplicative factor) also when $i = j$. Thus, it follows from Lemma 3.2 that

$$\begin{aligned} \mathbb{E}\left((\nu_n(S))^2\right) &= \sum_{i,j=1}^{M_n} q_{n,i}^{-1} q_{n,j}^{-1} E(Y(n,i)Y(n,j))(2\epsilon_n)^4 \\ &\leq C_1 \sum_{i,j=1}^{M_n} \epsilon_n^4 \epsilon_{l(i,j)}^{-a-\delta_{l(i,j)}} \leq C_2 \sum_{l=1}^{\infty} l^6 \epsilon_l^{2-a-\delta_l} < \infty \end{aligned} \tag{3.14}$$

is a bounded sequence (recall that $\delta_l \to 0$). Applying the Paley-Zygmund inequality (see [8, page 8]), (3.10) and (3.14) together guarantee that for some $b > 0$, $v > 0$

$$\mathbf{P}(b^{-1} \geq \nu_n(S) \geq b) \geq 2v > 0, \quad \forall n. \tag{3.15}$$

Similarly, for $h = 2 - a - \delta \in (0,2)$,

$$\begin{aligned} \mathbb{E}(\mathcal{G}_h(\nu_n)) &\leq C_3 \sum_{i,j=1}^{M_n} \frac{\mathbb{E}(Y(n,i)Y(n,j))}{q_{n,i}q_{n,j}} \int_{S(n,i)} \int_{S(n,j)} |x-y|^{-h}\, dx\, dy \\ &\leq C_4 \sum_{i,j=1}^{M_n} \epsilon_n^4 \epsilon_{l(i,j)}^{-a-h-\delta_{l(i,j)}} \leq C_5 \sum_{l=1}^{\infty} l^6 \epsilon_l^{2-h-a-\delta_l} < \infty \end{aligned} \tag{3.16}$$

is a bounded sequence. Thus we can find $d < \infty$ such that

$$\mathbf{P}(\mathcal{G}_h(\nu_n) \leq d) \geq 1 - v > 0, \quad \forall n. \tag{3.17}$$

Combined with (3.15) this shows that

$$\mathbf{P}(b^{-1} \geq \nu_n(S) \geq b,\, \mathcal{G}_h(\nu_n) \leq d) \geq v > 0, \quad \forall n. \tag{3.18}$$

Let $\mathcal{C}_n = \{\omega : b^{-1} \geq \nu_n(S) \geq b, \, \mathcal{G}_h(\nu_n) \leq d\}$ and set $\mathcal{C} = \limsup_n \mathcal{C}_n$. Then, (3.18) implies that

$$\mathbf{P}(\mathcal{C}) \geq v > 0. \tag{3.19}$$

Fixing $\omega \in \mathcal{C}$ there exists a subsequence $n_k \to \infty$ such that $\omega \in \mathcal{C}_{n_k}$ for all k. Due to the lower semicontinuity of $\mathcal{G}_h(\cdot)$, the set of non-negative measures ν on S such that $\nu(S) \in [b, b^{-1}]$ and $\mathcal{G}_h(\nu) \leq d$ is compact with respect to weak convergence. Thus, for $\omega \in \mathcal{C}$, the sequence $\nu_{n_k} = \nu_{n_k,\omega}$ has at least one weak limit ρ_ω which is a finite measure supported on $F(\omega)$, having positive mass and finite h-energy. This completes the proof of (3.8), hence that $\mathbf{P}(\mathcal{E}_1) > 0$. □

4. The coarse multifractal spectrum

Proof of Theorem 1.4: Fix $a \in (0,2)$ and let

$$C(\varepsilon, a) = \{x : \mu_\theta^w(D(x,\varepsilon)) \geq a\varepsilon^2 (\log \varepsilon)^2\}\,.$$

With $\tilde{\varepsilon}_n$ as in (2.9), the bound (2.13) yields for some $c_i = c_i(\delta) < \infty$ and any $\eta > 0$,

$$\mathbf{P}(\mathcal{L}eb(C(\tilde{\varepsilon}_n, a)) \geq \tilde{\varepsilon}_n^\eta) \leq c_1 \mathbb{E}|\mathcal{A}_n| \tilde{\varepsilon}_n^{2-\eta} \leq c_2 \tilde{\varepsilon}_n^{(1-10\delta)a-\eta}\,.$$

The Borel-Cantelli lemma and (2.10) then imply that

$$\liminf_{\varepsilon \to 0} \frac{\log \mathcal{L}eb(C(\varepsilon, a/(1-\delta)))}{\log \varepsilon} \geq a(1-10\delta)\,, \quad a.s.$$

Taking $\delta \to 0$ then yields the conclusion

$$\liminf_{\varepsilon \to 0} \frac{\log \mathcal{L}eb(C(\varepsilon, a))}{\log \varepsilon} \geq a\,, \quad a.s.$$

Turning to a complementary upper bound, fix $\delta > 0$ such that $a(1+\delta)^3 < 2$. Let $\epsilon_\delta = \epsilon\delta/(1+\delta)$, $C_\delta = C(\varepsilon/(1+\delta), a(1+\delta)^3)$ and $N(\epsilon)$ a (finite) maximal set of $x_i \in C_\delta$ such that $|x_i - x_j| > 2\epsilon_\delta$ for all $i \neq j$. Note that $\{D(x_i, \epsilon_\delta) : x_i \in N(\epsilon)\}$ are disjoint and if $x \in C_\delta$ then $D(x, \epsilon_\delta) \subset C(\varepsilon, a)$. Therefore,

$$\pi \epsilon_\delta^2 |N(\epsilon)| \leq \mathcal{L}eb(\cup_{x \in C_\delta} D(x, \epsilon_\delta)) \leq \mathcal{L}eb(C(\varepsilon, a))\,.$$

With $d(\epsilon) = \log |N(\varepsilon)| / \log(1/\epsilon)$, we thus see that

$$\liminf_{\epsilon \to 0} d(\epsilon) \leq 2 - \limsup_{\varepsilon \to 0} \frac{\log \mathcal{L}eb(C(\varepsilon, a))}{\log \varepsilon}\,. \tag{4.1}$$

Let

$$\mathsf{CThick}_{\geq a} = \{x \in D(0,1) : \liminf_{\varepsilon \to 0} \frac{\mu_\theta^w(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} \geq a\}\,, \tag{4.2}$$

and

$$\mathsf{CThick}_{\gamma, \geq a} = \{x \in D(0,1) : \inf_{\epsilon \leq \gamma} \frac{\mu_\theta^w(D(x,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} \geq a\}.$$

The sets $\mathsf{CThick}_{\gamma,\geq a}$ are monotone nonincreasing in γ and

$$\mathsf{CThick}_{\geq a(1+\delta)^4} \subseteq \bigcup_n \mathsf{CThick}_{\gamma_n,\geq a(1+\delta)^3} \tag{4.3}$$

for any $\gamma_n \to 0$. Recall that $\mathcal{S}_\epsilon := \{D(x_i, 3\epsilon_\delta) \,:\, x_i \in N(\epsilon)\}$, forms a cover of C_δ, so à fortiori it is also a cover of $\mathsf{CThick}_{\varepsilon/(1+\delta),\geq a(1+\delta)^3}$. Fixing $\epsilon_n \downarrow 0$ it follows from (4.3) that $\cup_{n\geq m}\mathcal{S}_{\epsilon_n}$ is a cover of $\mathsf{CThick}_{\geq a(1+\delta)^4}$ by sets of maximal diameter $6\epsilon_m$. Hence, the η-Hausdorff measure of $\mathsf{CThick}_{\geq a(1+\delta)^4}$ is finite for any η such that

$$\sum_{n=1}^{\infty} |N(\epsilon_n)|\epsilon_n^{\eta} = \sum_{n=1}^{\infty} \epsilon_n^{\eta-d(\epsilon_n)} < \infty\,,$$

that is, whenever $\eta > \liminf_{\epsilon\to 0} d(\epsilon)$. Consequently, by (4.1)

$$\dim(\mathsf{CThick}_{\geq a(1+\delta)^4}) \leq \liminf_{\epsilon\to 0} d(\epsilon) \leq 2 - \limsup_{\varepsilon\to 0} \frac{\log \mathcal{L}eb(C(\varepsilon,a))}{\log \varepsilon}\,. \tag{4.4}$$

Taking $\delta \to 0$ and using (1.14) yields that

$$\limsup_{\varepsilon\to 0} \frac{\log \mathcal{L}eb(C(\varepsilon,a))}{\log \varepsilon} \leq a\,, \quad a.s.\,,$$

as needed to complete the proof. ☐

5. The Erdős-Taylor conjecture

We present here the generalization of Theorem 1.1 alluded to in the introduction. Recall that a random walk in $\mathbf{Z}^2$ is *aperiodic* if the increments are not supported on a proper subgroup of $\mathbf{Z}^2$.

Theorem 5.1. *Let $S_n = \sum_{i=1}^n X_i$ be an aperiodic random walk with i.i.d. increments $X_i \in \mathbf{Z}^2$ that satisfy $\mathbb{E}X = 0$ and $\mathbb{E}|X|^m < \infty$ for all $m < \infty$. Denote by $\Gamma = \mathbb{E}XX'$ the covariance matrix of the increments, and write $\pi_\Gamma := 2\pi(\det\Gamma)^{1/2}$. Consider the local time,*

$$T_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{S_k = x\}} \quad (x \in \mathbf{Z}^2)$$

and its maximum $T_n^* = \max_{x\in\mathbf{Z}^2} T_n(x)$. Let $M(n,\alpha)$ denote the number of points in the set $\{x : T_n(x) \geq \alpha(\log n)^2\}$. Then,

$$\lim_{n\to\infty} \frac{T_n^*}{(\log n)^2} = \pi_\Gamma^{-1}\,, \quad a.s.\,, \tag{5.1}$$

and for $\alpha \in (0, \pi_\Gamma^{-1}]$,

$$\lim_{n\to\infty} \frac{\log M(n,\alpha)}{\log n} = 1 - \alpha\pi_\Gamma\,, \quad a.s. \tag{5.2}$$

Moreover, any (random) sequence $\{x_n\}$ *in* $\mathbf{Z}^2$ *such that* $T_n(x_n)/T_n^* \to 1$*, must satisfy*

$$\lim_{n\to\infty} \frac{\log|x_n|}{\log n} = \frac{1}{2} \quad a.s. \tag{5.3}$$

(Note that for simple random walk $\Gamma = \frac{1}{2}I$, so $\pi_\Gamma = \pi$).

Proof of Theorem 5.1: We start by proving the lower bound

$$\liminf_{n\to\infty} \frac{T_n^*}{(\log n)^2} \geq \frac{1}{2\pi(\det\Gamma)^{1/2}} = \pi_\Gamma^{-1}, \quad a.s.. \tag{5.4}$$

(For the case of simple random walk, this will prove the Erdős-Taylor conjecture, as the upper bound is already in [7, (3.11)].) Our approach is to use Theorem 1.3 together with the strong approximation results of [6] and [11].

Fixing $\delta > 0$, it follows from (1.8) that a.s.

$$\liminf_{\varepsilon\to 0} \sup_{|z|<1} \frac{\mu_{\bar\theta}^w(D(z,\varepsilon))}{\varepsilon^2|\log\varepsilon|^2} \geq \sup_{|z|<1} \liminf_{\varepsilon\to 0} \frac{\mu_{\bar\theta}^w(D(z,\varepsilon))}{\varepsilon^2|\log\varepsilon|^2} \geq 2-\delta/2\,.$$

Hence,

$$\lim_{\varepsilon\to 0} \mathbf{P}\Big(\sup_{|z|<1} \frac{\mu_{\bar\theta}^w(D(z,\varepsilon))}{\varepsilon^2|\log\varepsilon|^2} \geq 2-\delta\Big) = 1\,.$$

Since $\mathbf{P}(\bar\theta \leq 1) > 0$, it follows that for some $\bar p_0 > 0$, $\epsilon_1 > 0$ and all $\varepsilon < \epsilon_1$,

$$\mathbf{P}\Big(\sup_{|z|<1} \frac{\mu_1^w(D(z,\varepsilon))}{\varepsilon^2|\log\varepsilon|^2} \geq 2-\delta\Big) \geq 3\bar p_0\,.$$

In particular, fix $\eta > 0$ and let $\varepsilon_n = n^{\eta-1/2}$. Then for large n,

$$\mathbf{P}\Big(\sup_{|z|<1} \frac{\mu_1^w(D(z,\varepsilon_n))}{\varepsilon_n^2|\log\varepsilon_n|^2} \geq 2(1-\delta)\Big) \geq 3\bar p_0. \tag{5.5}$$

Since, by Lévy's modulus of continuity,

$$\lim_{n\to\infty} \mathbf{P}\Big(\sup_{0\leq t\leq 1} |w_{[nt]/n} - w_t| \geq \delta\varepsilon_n\Big) = 0\,,$$

it follows that for large n,

$$\mathbf{P}\Big(\sup_{|z|<1} \frac{\sum_{j=1}^n \mathbf{1}_{|w_{j/n}-z|<(1+\delta)\varepsilon_n}}{n\varepsilon_n^2|\log\varepsilon_n|^2} \geq 2(1-\delta)^2\Big) \geq 2\bar p_0\,. \tag{5.6}$$

By Einmahl's [6, Theorem 1] multidimensional extension of the Komlós-Major-Tusnády [11] strong approximation theorem, we may, for each n, construct $\{S_k\}_{k=1}^n$ and $\{w_t\}_{0\leq t\leq 1}$ on the same probability space so that

$$\lim_{n\to\infty} \mathbf{P}\Big(\max_{k=1,\dots,n} |w_{k/n} - \frac{1}{\sqrt{n}}\Gamma^{-1/2}S_k| > \delta\varepsilon_n\Big) = 0.$$

(For the case where S_n is a $\mathbf{Z}^2$ valued simple random walk, the original construction of KMT [11] suffices, since rotating the axes by $\pi/4$, one may view S_n as two independent one-dimensional simple random walks of step size $1/\sqrt{2}$.) Combining the above with (5.6), it follows that for large n,

$$\mathbf{P}\Big(\sup_{y\in \mathbb{R}^2} \frac{\sum_{j=1}^n \mathbf{1}_{|\Gamma^{-1/2}S_j - y| < \sqrt{n}(1+2\delta)\varepsilon_n}}{n\varepsilon_n^2 |\log \varepsilon_n|^2} \geq 2(1-\delta)^3\Big) \geq \bar{p}_0. \tag{5.7}$$

The number of lattice points in the ellipse $\{z : |\Gamma^{-1/2}z - y| < \sqrt{n}(1+2\delta)\varepsilon_n\}$ is less than $\pi(\det\Gamma)^{1/2}n(1+2\delta)^3\varepsilon_n^2$, so by the pigeonhole principle,

$$T_n^* \geq \sup_{y\in\mathbb{R}^2} \frac{\sum_{j=1}^n \mathbf{1}_{|\Gamma^{-1/2}S_j - y| < \sqrt{n}(1+2\delta)\varepsilon_n}}{\pi(\det\Gamma)^{1/2}n(1+2\delta)^3\varepsilon_n^2}\,.$$

Since $|\log\varepsilon_n| = (\frac{1}{2}-\eta)\log n$, we infer that for large n,

$$\mathbf{P}\Big(T_n^* \leq \frac{(1-\delta)^3(1-2\eta)^2(\log n)^2}{(1+2\delta)^3\pi_\Gamma}\Big) \leq 1-\bar{p}_0\,.$$

Since a a path of length n contains $[n^\delta]$ disjoint segments of length $[n^{1-\delta}]$, using independence of increments we deduce for large enough n that

$$\begin{aligned} &\mathbf{P}\Big(T_n^* \leq \frac{(1-\delta)^6(1-2\eta)^2(\log n)^2}{(1+2\delta)^3\pi_\Gamma}\Big) \\ &\leq \Big[\mathbf{P}\Big(T_{[n^{1-\delta}]}^* \leq \frac{(1-\delta)^3(1-2\eta)^2(\log[n^{1-\delta}])^2}{(1+2\delta)^3\pi_\Gamma}\Big)\Big]^{[n^\delta]} \leq (1-\bar{p}_0)^{[n^\delta]}\,. \end{aligned}$$

An application of the Borel-Cantelli lemma followed by taking the limit as $\delta,\eta \downarrow 0$ completes the proof of (5.4).

To establish (5.1), it remains to verify the upper bound

$$\limsup_{n\to\infty} \frac{T_n^*}{(\log n)^2} \leq \frac{1}{\pi_\Gamma}\,, \quad a.s.. \tag{5.8}$$

If $\{S_n\}$ is *strongly aperiodic*, that is, if the increments are not supported on a coset of a proper subgroup of $\mathbf{Z}^2$, then the local CLT in Spitzer [20, Section 7, P9] implies that

$$\sum_{k=0}^n \mathbf{P}[S_k = 0] \sim \frac{\log n}{\pi_\Gamma} \;\text{ as } n\to\infty\,. \tag{5.9}$$

In fact, our standing aperiodicity assumption suffices to get (5.9):
Let $h := \text{g.c.d.}\{n : \mathbf{P}(S_n = 0) > 0\}$. From Spitzer [20, Section 5, P1] it follows that either S_{nh} is strongly aperiodic (in which case (5.9) holds) or it is periodic. Using Spitzer [20, Section 7, P1], we may infer that there exists a 2×2 integer matrix A, and a strongly aperiodic random walk $\{\widetilde{S}_n\}$ in $\mathbf{Z}^2$, such that $S_{nh} = A\widetilde{S}_n$ for all $n\geq 1$. (See the discussion in [10, pages 659-660] for a similar argument). By the remark following [20, Section 5, P1], the index of the subgroup $A\mathbf{Z}^2$ in $\mathbf{Z}^2$ is h; on the other hand, this index equals $|\det A|$ by the counting argument in the proof of [20, Section 7, P2] (cover $\mathbf{Z}^2$ by h cosets of $A\mathbf{Z}^2$, and use the transformation of volumes by the factor $|\det A|$). Denote by Γ and $\widetilde{\Gamma}$ the covariance matrices for the increments of

$\{S_n\}$ and $\{\widetilde{S}_n\}$, respectively. Since $h\Gamma = A\widetilde{\Gamma}A'$, the determinants of Γ and $\widetilde{\Gamma}$ coincide, and (5.9) follows.

Applying [2, Theorem 8.7.3] for the renewal sequence $u_n = \mathbf{P}(S_n = 0)$, we deduce from (5.9) that for all large n,

$$\mathbf{P}\Big(\forall k \in [1,n),\ S_k \neq 0\Big) > \frac{(1-\delta)\pi_\Gamma}{\log n}\,. \tag{5.10}$$

By the strong Markov property,

$$\begin{aligned}\mathbf{P}[T_n(0) \geq \alpha(\log n)^2] &< \Big(1 - \frac{(1-\delta)\pi_\Gamma}{\log n}\Big)^{\alpha(\log n)^2} \\ &\leq e^{-(1-\delta)\alpha\pi_\Gamma \log n} = n^{-(1-\delta)\alpha\pi_\Gamma}. \end{aligned}\tag{5.11}$$

Hence,

$$\begin{aligned}&\mathbf{P}(T_n^* \geq \alpha(\log n)^2) \leq \mathbb{E}M(n,\alpha) \\ &\leq \sum_{k=1}^n \mathbf{P}[\forall j \in [1,k),\ S_k \neq S_j,\ \ T_n(S_k) \geq \alpha(\log n)^2] \\ &< n^{1-(1-\delta)\alpha\pi_\Gamma}\,. \end{aligned}\tag{5.12}$$

If $\alpha > \pi_\Gamma^{-1}$ then by taking $\delta > 0$ small enough, we ensure that the right-hand-side of (5.12) is summable on the subsequence $n_m = 2^m$. By the Borel-Cantelli lemma we get (5.8) for this subsequence, hence by interpolation for all n.

The proof of (5.2) is very similar: Fix $a < 2$, $\eta > 0$ and $\varepsilon_n = n^{\eta-1/2}$. Recall the definition of $N(\varepsilon)$ from Section 4 for $\delta > 0$ small enough. The argument of that section shows that for some $\bar{p}_1 > 0$ and all n large enough,

$$\mathbf{P}\Big(\bar{\theta} \leq 1, |N(\varepsilon_n)| \geq \varepsilon_n^{a(1+\delta)^5-2}\Big) \geq 3\bar{p}_1\,.$$

On this event we can find in each $N(\varepsilon_n)$ a subset of at least $\delta^3\varepsilon_n^{a(1+\delta)^5-2}$ points that are $3\varepsilon_n$ separated. By Lévy's modulus of continuity, the multidimensional strong approximation of [6, Theorem 1] and the pigeonhole principle, we may infer that for $\alpha = (a/2-\delta')\pi_\Gamma^{-1}$, some $\delta'(\eta,\delta) > 0$ such that $\delta' \downarrow 0$ when $\delta \vee \eta \downarrow 0$, and all large n,

$$\mathbf{P}[M(n,\alpha) \geq n^{1-\alpha\pi_\Gamma - 2\delta'}] \geq \bar{p}_1\,.$$

The lower bound follows by partitioning a path of length n to n^δ segments of length $n^{1-\delta}$ each, using independence of increments, the Borel-Cantelli lemma and considering $\delta, \eta \downarrow 0$.

The corresponding upper bound follows from (5.12) by Markov's inequality and an application of the Borel-Cantelli lemma along the subsequence n_m.

Finally, for $\{x_n\}$ satisfying $T_n(x_n)/T_n^* \to 1$, for any $\eta > 0$, a.s.

$$|x_n| \leq \max_{k=1}^n |S_k| \leq n^{1/2+\eta}\,,$$

for all n large enough. On the other hand, the inequality (5.11) implies that

$$\mathbf{P}\Big(\sup_{|y|\le n^{1/2-\eta}} T_n(y) \ge \alpha(\log n)^2\Big) \le Cn^{1-2\eta-(1-\delta)\alpha\pi_\Gamma}\,,$$

which is summable on the subsequence $n_m = 2^m$ if $\alpha\pi_\Gamma > 1-2\eta$ and δ is small enough. Invoking the Borel-Cantelli lemma and the monotonicity of $n \mapsto \sup_{|y|\le n^{1/2-\eta}} T_n(y)$ it follows that a.s.

$$\limsup_{n\to\infty} \frac{\sup_{|y|\le n^{1/2-\eta}} T_n(y)}{(\log n)^2} \le (1-2\eta)\pi_\Gamma^{-1}\,,$$

so that (5.3) follows from (5.4). □

6. From excursions to occupation times

Recall that N_k^x denotes the number of excursions from $\partial D(x,\epsilon_{k-1})$ to $\partial D(x,\epsilon_k)$ prior to $\bar\theta$. Fixing $a \in (0,2)$ and $n_k = 3ak^2\log k$ we call $x \in S$ **lower k-successful** if $N_k^x \ge n_k - k$, and $x \in S$ is called **lower m-perfect** if it is lower k-successful for all $k = 2,\dots,m$. Recall that if $x \in S$ is m-perfect then also

$$n_k - k \le N_k^x \le n_k + k\,, \qquad \forall k = 2,\dots,m,$$

and hence x is lower m-perfect.

With $h(\varepsilon) := \varepsilon^2|\log\varepsilon|^2$, the following lemma gives the lower bound in Lemma 3.1.

Lemma 6.1. *There exists a $\delta(\epsilon) = \delta(\epsilon,\omega) \to 0$ a.s. such that for all m and all $x \in S$, if x is lower m-perfect then*

$$(a-\delta(\epsilon))h(\epsilon) \le \mu_{\bar\theta}^w(D(x,\epsilon)), \qquad \forall\epsilon \ge \epsilon_m. \tag{6.1}$$

Proof of Lemma 6.1: Let $\delta_k = \epsilon_k/k^6$ and let $\mathcal{D}_k$ be a δ_k-net of points in S. Let

$$\epsilon'_k = \epsilon_k e^{1/k^6}, \quad \epsilon''_{k-1} = \epsilon_{k-1}e^{-1/k^6},$$

so that

$$\epsilon'_k \ge \epsilon_k + \delta_k, \quad \epsilon''_{k-1} \le \epsilon_{k-1} - \delta_k. \tag{6.2}$$

We will say that a point $x' \in \mathcal{D}_k$ is **successful** if there are at least $n_k - k$ excursions from $\partial D(x',\epsilon''_{k-1})$ to $\partial D(x',\epsilon'_k)$ prior to $\bar\theta$. Let

$$\epsilon_{k,j} = \epsilon_k e^{-j/k}\,,\ j = 0,1,\dots,3k\log(k+1),$$

and let $\epsilon'_{k,j} = \epsilon_{k,j}e^{-2/k^3} = \epsilon'_k e^{-j/k}e^{-2/k^3-1/k^6}$. We now derive Lemma 6.1 from the following lemma.

Lemma 6.2. *There exists a $\delta(\epsilon) = \delta(\epsilon,\omega) \to 0$ a.s. such that for all k and $x' \in \mathcal{D}_k$, if x' is successful then*

$$(a-\delta(\epsilon'_{k,j}))h(\epsilon'_{k,j}) \le \mu_{\bar\theta}^w(D(x',\epsilon'_{k,j})), \quad \forall j = 0,1,\dots,3k\log(k+1). \tag{6.3}$$

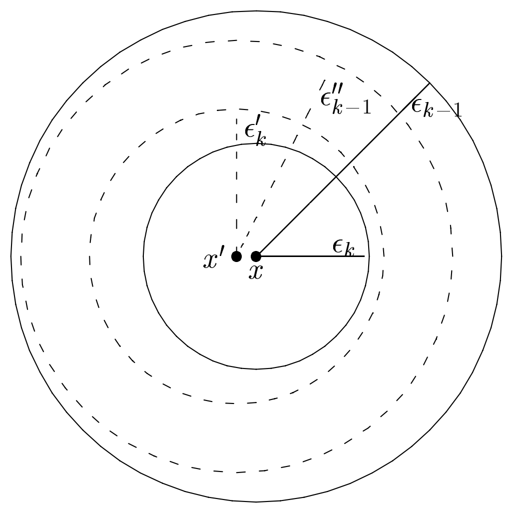

Figure 1.

For assume that Lemma 6.2 holds, and let $x \in S$ be lower m-perfect. For any $k \leq m$ we can find $x' \in \mathcal{D}_k$ with $|x - x'| \leq \delta_k$. Then, $D(x, \epsilon_k) \subseteq D(x', \epsilon'_k)$ and $D(x', \epsilon''_{k-1}) \subseteq D(x, \epsilon_{k-1})$, see (6.2) and Figure 1, so that the fact that x is lower k-successful implies that x' is successful. Thus (6.3) holds by Lemma 6.2. One easily checks that $\delta_k + \epsilon'_{k,j} \leq \epsilon_{k,j}$ for all j and any large k, implying that $D(x', \epsilon'_{k,j}) \subseteq D(x, \epsilon_{k,j})$. This, together with (6.3) and

$$h(\epsilon_{k,j}) \leq (1 + 6/k^3) h(\epsilon'_{k,j})$$

then shows that for all $j = 0, 1, \dots, 3k \log(k+1)$,

$$(a - \delta(\epsilon'_{k,j}) - 13/k^3) h(\epsilon_{k,j}) \leq \mu^w_{\bar\theta}(D(x, \epsilon_{k,j})) \,. \tag{6.4}$$

Now for any $\epsilon_{k+1} \leq \epsilon \leq \epsilon_k$, let j be such that $\epsilon_{k,j+1} \leq \epsilon \leq \epsilon_{k,j}$. Then,

$$\frac{\mu^w_{\bar\theta}(D(x,\epsilon))}{h(\epsilon)} \geq \frac{\mu^w_{\bar\theta}(D(x,\epsilon_{k,j+1}))}{h(\epsilon_{k,j})} \geq \frac{\mu^w_{\bar\theta}(D(x,\epsilon_{k,j+1}))}{h(\epsilon_{k,j+1})} (1 - 2/k) \;, \tag{6.5}$$

and Lemma 6.1 follows from (6.4) and (6.5).

Proof of Lemma 6.2: Suppose that $x' \in \mathcal{D}_k$ is successful. Then there are at least $n'_k = n_k - k$ excursions between $\partial D(x', \epsilon'_k)$ and $\partial D(x', \epsilon''_{k-1})$, where $n'_k \to \infty$ as $k \to \infty$. Let $\tau_{l,k,j}$ denote the occupation measure of $D(x', \epsilon'_{k,j}) \subset D(x', \epsilon'_k)$ during the l-th excursion. Note that the $\tau_{l,k,j}$ are i.i.d. and

$$\begin{aligned} P_{x',k,j} &:= \mathbf{P}\left(\int_0^{\bar\theta} \mathbf{1}_{\{w_t \in D(x', \epsilon'_{k,j})\}} dt \leq a(1 - 2/\log k) h(\epsilon'_{k,j}), x' \text{ is successful} \right) \\ &\leq \mathbf{P}\left(\sum_{l=1}^{n'_k} \tau_{l,k,j} \leq a(1 - 2/\log k) h(\epsilon'_{k,j}) \right) . \end{aligned}$$

Let

$$K_k = \log(\epsilon''_{k-1}/\epsilon'_k) = 3\log k - 2/k^6.$$

With k large enough, using Stirling's approximation for $\log\epsilon_k = \log\epsilon_1 - 3\log k!$,

$$a(1-2/\log k)|\log\epsilon'_{k,j}|^2 \le (1-1/\log k)K_k n'_k\,.$$

Hence,

$$P_{x',k,j} \le \mathbf{P}\left(\frac{1}{n'_k}\sum_{l=1}^{n'_k}\frac{\tau_{l,k,j}}{K_k{\epsilon'_{k,j}}^2} \le 1-1/\log k\right)\,. \tag{6.6}$$

Define $\widehat{\tau}_{l,k,j} := \frac{\tau_{l,k,j}}{K_k{\epsilon'_{k,j}}^2}$. Then, a substitution in Lemma 2.1 with $r_1 = \epsilon'_{k,j}$, $r_2 = \epsilon'_k$, $r_3 = \epsilon''_{k-1}$, reveals that for all k large enough,

$$\mathbb{E}(\widehat{\tau}_{l,k,j}) = 1\,, \qquad \mathbb{E}(\widehat{\tau}^2_{l,k,j}) \le 10\,, \tag{6.7}$$

so that, with $\widetilde{\tau}_{l,k,j} := \widehat{\tau}_{l,k,j} - \mathbb{E}(\widehat{\tau}_{l,k,j})$ we have

$$P_{x',k,j} \le \mathbf{P}\left(\frac{1}{n'_k}\sum_{l=1}^{n'_k}\widetilde{\tau}_{l,k,j} \le -1/\log k\right)\,. \tag{6.8}$$

Since $\widetilde{\tau}_{l,k,j} \ge -1$, it follows that for all $0<\theta<1$,

$$\mathbb{E}(e^{-\theta\widetilde{\tau}_{l,k,j}}) \le 1 + 2\theta^2\mathbb{E}(\widetilde{\tau}^2_{l,k,j}) \le 1+20\theta^2 \le e^{20\theta^2}\,.$$

Taking $\theta = \lambda/(a\log k)$, a standard application of Chebyscheff's inequality then shows that for some $\lambda = \lambda(a) > 0$, $C_1 < \infty$ and all $x' \in \mathcal{D}_k$, k, j

$$P_{x',k,j} \le C_1 e^{-\lambda k^2/\log k}\,. \tag{6.9}$$

Since $|\mathcal{D}_k| \le e^{C_2 k\log k}$ for some $C_2 < \infty$ and all k, it follows that

$$\sum_{k=1}^{\infty}\sum_{j=0}^{3k\log(k+1)}\sum_{x'\in\mathcal{D}_k} P_{x',k,j} \le 3C_1\sum_{k=1}^{\infty} k^2 e^{C_2 k\log k}e^{-\lambda k^2/\log k} < \infty\,.$$

The Borel-Cantelli lemma completes the proof of Lemma 6.2. □

We turn to the upper bound in Lemma 3.1. The situation here is quite similar to the lower bound. A point $x \in S$ is **upper k-successful** if $N_k^x \le n_k + k$, and $x \in S$ is **upper m-perfect** if it is upper k-successful for all $k = 2,\dots,m$. Since every m-perfect $x\in S$ is upper m-perfect, the following lemma gives the upper bound in Lemma 3.1.

Lemma 6.3. *There exists a $\delta(\epsilon) = \delta(\epsilon,\omega) \to 0$ a.s. such that for all m and all $x \in S$, if x is upper m-perfect then*

$$\mu_\theta^w(D(x,\epsilon)) \le (a+\delta(\epsilon))h(\epsilon), \qquad \forall\epsilon \ge \epsilon_m. \tag{6.10}$$

Let now

$$\bar{\epsilon}'_k = \epsilon_k e^{-2/k^6}, \quad \bar{\epsilon}''_{k-1} = \epsilon_{k-1} e^{1/k^6},$$

so that

$$\bar{\epsilon}'_k \leq \epsilon_k - \delta_k, \quad \bar{\epsilon}''_{k-1} \geq \epsilon_{k-1} + \delta_k. \tag{6.11}$$

We now say that $x' \in \mathcal{D}_k$ is **u-successful** if there are at most $n_k + k$ excursions from $\partial D(x', \bar{\epsilon}''_{k-1})$ to $\partial D(x', \bar{\epsilon}'_k)$ prior to $\bar{\theta}$. We can derive Lemma 6.3 from the following lemma.

Lemma 6.4. *There exists a $\delta(\epsilon) = \delta(\epsilon, \omega) \to 0$ a.s. such that for all k and $x' \in \mathcal{D}_k$, if x' is u-successful then*

$$\mu^w_{\bar{\theta}}(D(x', \epsilon'_{k,j})) \leq (a + \delta(\epsilon'_{k,j}))h(\epsilon'_{k,j}), \quad \forall j = 1, \dots, 3k \log(k+1). \tag{6.12}$$

Note that here we take $j \geq 1$ to insure that $\epsilon'_{k,j} \leq \bar{\epsilon}'_k$. As with the lower bound, (6.12) leads, for $x \in S$ which is upper m-perfect, to

$$\mu^w_{\bar{\theta}}(D(x, \epsilon_{k,j})) \leq (a + \delta(\epsilon'_{k,j}) + 13/k^3)h(\epsilon_{k,j})\,, \tag{6.13}$$

for all $j = 1, \dots, 3k \log(k+1)$. Lemma 6.3 then follows as with the lower bound, noting that we also have (6.13) for $j = 0$ since $\epsilon_{k,0} = \epsilon_{k-1,3(k-1)\log(k)}$.

Since $0 \notin D(x, \epsilon_1)$ for all $x \in S$, with $n''_k = n_k + k$, the proof of Lemma 6.4, in analogy to that of Lemma 6.2, comes down to bounding

$$\mathbf{P}\left(\sum_{l=1}^{n''_k} \widetilde{\tau}_{l,k,j} \geq n''_k / \log k\right) \leq e^{-\lambda n''_k / \log k} \left(E\left(e^{\lambda \widetilde{\tau}_{1,k,j}}\right)\right)^{n''_k}.$$

Noting that, by (2.2), for some $C < \infty$, all $\lambda > 0$ small and k large enough,

$$E\left(e^{\lambda \widetilde{\tau}_{1,k,j}}\right) = 1 + \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} E\left(\widetilde{\tau}^n_{1,k,j}\right) \leq 1 + \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} E\left(\widehat{\tau}^n_{1,k,j} + 1\right) \leq 1 + C\lambda^2\,,$$

the proof of Lemma 6.4 now follows as in the proof of Lemma 6.2.

This concludes the proof of Lemma 3.1. ☐

7. First moment estimates

Fixing $\zeta = 3a > 0$, recall that $\epsilon_1 = 1/8$, $\epsilon_k = k^{-3}\epsilon_{k-1}$ and $n_k = \zeta k^2 \log k$ for $k \geq 2$. Recall also that $N^x_k(\rho)$ denotes the number of excursions from $\partial D(x, \epsilon_{k-1})$ to $\partial D(x, \epsilon_k)$ prior to $\sigma_{x,\rho} = \inf\{t \geq 0 : w_t \in \partial D(x, \rho)\}$, and $x \in S$ is called n-perfect when $|N^x_k(1/2) - n_k| \leq k$ and $|N^x_k(2) - n_k| \leq k$ for $k = 2, \dots, n$. Throughout we let $\bar{\sigma}_{x,\rho}$ denote the corresponding hitting times for $\{w_{\sigma_{x,1/2}+t}, t \geq 0\}$ and set $\epsilon_0 = 1/2$. Our next lemma provides the lower bound (3.11) on $\mathbf{P}(x$ is n-perfect$)$ as well as a uniform (for $x \in S$) complementary upper bound.

Lemma 7.1. *For all $n \geq 2$ and some $\delta_n \to 0$,*

$$q_n := \mathbf{P}\left(n_k - k \leq N_k^x(1/2) \leq n_k + k; 2 \leq k \leq n | \sigma_{x,\epsilon_1} < \sigma_{x,1/2}\right)$$

$$= (n!)^{-\zeta-\delta_n} . \tag{7.1}$$

Moreover, for some $c > 0$ and all $x \in S$,

$$q_n \geq \mathbf{P}(x \text{ is } n\text{-perfect}) \geq c q_n . \tag{7.2}$$

Proof of Lemma 7.1: Recall that eventually $n_k - k \geq 1$, and that $0 \notin D(x, \epsilon_1)$ for all $x \in S$. Hence, for $x \in S$ to be n-perfect, necessarily $\sigma_{x,\epsilon_1} < \sigma_{x,1/2}$ resulting with the upper bound in (7.2). Turning to the lower bound, consider the event $\{\bar{\sigma}_{x,2} < \bar{\sigma}_{x,\epsilon_1}\}$ which guarantees that $N_k^x(2) = N_k^x(1/2)$ for all k. By the radial symmetry of the events considered and the strong Markov property of Brownian motion, the event $\{\bar{\sigma}_{x,2} < \bar{\sigma}_{x,\epsilon_1}\}$ is independent of both $\{N_k^x(1/2); k \geq 2\}$ and $\{\sigma_{x,\epsilon_1} < \sigma_{x,1/2}\}$, with $\mathbf{P}(\bar{\sigma}_{x,2} < \bar{\sigma}_{x,\epsilon_1}) = \bar{p}_3 > 0$ independent of the value of $x \in S$. Note that $\mathbf{P}(\sigma_{x,\epsilon_1} < \sigma_{x,1/2})$, for $x \in S$, is a monotone decreasing function of $|x|$ that is positive for all such x, hence $\bar{p}_4 := \inf_{x \in S} \mathbf{P}(\sigma_{x,\epsilon_1} < \sigma_{x,1/2}) > 0$. The lower bound of (7.2) thus follows (with $c = \bar{p}_3 \bar{p}_4 > 0$).

Consider the birth-death Markov chain $\{X_l\}$ on $\{0, 1, 2, \dots\}$, starting at $X_0 = 1$, having 0 as its absorbing state and the transition probabilities for $k = 1, 2, \dots$

$$p_k := P(X_l = k+1 | X_{l-1} = k) = 1 - P(X_l = k-1 | X_l = k)$$

$$= \frac{\log(\epsilon_{k-1}/\epsilon_k)}{\log(\epsilon_{k-1}/\epsilon_{k+1})} . \tag{7.3}$$

Let $L_1 = 1$ and for each $k \geq 2$,

$$L_k = \sum_{l=0}^{\infty} 1_{\{X_l = k-1, X_{l+1} = k\}} ,$$

denote the number of transitions of $\{X_l\}$ from state $k-1$ to state k. Observe that p_k is exactly the probability that a path of w_t starting at $\partial D(x, \epsilon_k)$ will hit $\partial D(x, \epsilon_{k+1})$ prior to hitting $\partial D(x, \epsilon_{k-1})$, with (X_l, X_{l+1}) recording the order of excursions the path makes between the sets $\{\partial D(x, \epsilon_k), k \geq 1\}$ prior to $\sigma_{x,1/2}$. By the radial symmetry and the strong Markov property of Brownian motion, q_n of (7.1) is independent of $x \in S$. Moreover, fixing $x \in S$, conditioned upon $\{\sigma_{x,\epsilon_1} < \sigma_{x,1/2}\}$, the law of $\{N_k^x(1/2), k \geq 2\}$ is exactly that of $\{L_k, k \geq 2\}$.

Conditional on $L_k = \ell_k \geq 1$ we have the representation

$$L_{k+1} = \sum_{i=1}^{\ell_k} Y_i , \tag{7.4}$$

where the Y_i are independent identically distributed (geometric) random variables with

$$\mathbf{P}(Y_i = j) = (1 - p_k) p_k^j , \qquad j = 0, 1, 2, \dots \tag{7.5}$$

Consequently, $\{L_k, k \geq 1\}$ is a Markov chain on $\mathbf{Z}_+$ with transition probabilities

$$\mathbf{P}\left(L_{k+1} = m \,\middle|\, L_k = \ell+1\right) = \binom{m+\ell}{m} p_k^m (1-p_k)^{\ell+1}, \tag{7.6}$$

for $k \geq 1$, $m, \ell \geq 0$ and $\mathbf{P}(L_{k+1} = 0 | L_k = 0) = 1$ for all $k \geq 2$. We thus deduce that

$$\begin{aligned} q_n &= \mathbf{P}\left(n_k - k \leq L_k \leq n_k + k\,;\, 2 \leq k \leq n\right) \\ &= \sum_{\substack{\ell_2,\dots,\ell_n \\ |\ell_k - n_k| \leq k}} \prod_{k=1}^{n-1} \mathbf{P}\left(L_{k+1} = \ell_{k+1} \,\middle|\, L_k = \ell_k\right) \end{aligned} \tag{7.7}$$

(where $\ell_1 = 1$). The number of vectors $(\ell_2, \dots, \ell_n)$ considered in (7.7) is at least $n!$ and at most $3^n n!$. Since $n^{-1} \log n! \to \infty$ and for some $\eta_n \to 0$

$$\prod_{k=2}^{n} \log(k) = (n!)^{\eta_n} \ ,$$

we see that the estimate (7.1) on q_n is a direct consequence of (7.7) and the next lemma.

Lemma 7.2. *For some $C = C(\zeta) < \infty$ and all $k \geq 2$, $|m - n_{k+1}| \leq k+1$, $|\ell + 1 - n_k| \leq k$,*

$$C^{-1} \frac{k^{-\zeta}}{\sqrt{k^2 \log k}} \leq \mathbf{P}\left(L_{k+1} = m \,\middle|\, L_k = \ell+1\right) \leq C \frac{k^{-\zeta}}{\sqrt{k^2 \log k}} \,. \tag{7.8}$$

Proof of Lemma 7.2: It suffices to consider $k \gg 1$ in which case from (7.3),

$$p_k = \frac{\log(k)}{\log(k) + \log(k+1)} = \frac{1}{2} - O\left(\frac{1}{k \log k}\right), \tag{7.9}$$

and since $n_k - 2k \to \infty$, the binomial coefficient in (7.6) is well approximated by Stirling's formula

$$m! = \sqrt{2\pi} m^m e^{-m} \sqrt{m} (1 + o(1)) \ .$$

With $n_k = \zeta k^2 \log k$ it follows that for some $C_1 < \infty$ and all k large enough, if $|m - n_{k+1}| \leq 2k$, $|\ell - n_k| \leq 2k$ then

$$\left|\frac{m}{\ell} - 1 - \frac{2}{k}\right| \leq \frac{C_1}{k \log k} \ . \tag{7.10}$$

Hereafter, we use the notation $f \sim g$ if f/g is bounded and bounded away from zero as $k \to \infty$, uniformly in $\{m : |m - n_{k+1}| \leq 2k\}$ and $\{\ell : |\ell - n_k| \leq 2k\}$. We then have by (7.6) and the preceeding observations that

(7.11)

$$\mathbf{P}\left(L_{k+1} = m \,\middle|\, L_k = \ell+1\right) \sim \frac{(m+\ell)^{m+\ell}}{\sqrt{\ell}\, \ell^\ell m^m} p_k^m (1-p_k)^\ell \sim \frac{\exp(-\ell I(\frac{m}{\ell}, p_k))}{\sqrt{k^2 \log k}} \ ,$$

where

$$I(\lambda,p) = -(1+\lambda)\log(1+\lambda) + \lambda\log\lambda - \lambda\log p - \log(1-p)\ .$$

The function $I(\lambda,p)$ and its first order partial derivatives vanish at $(1,1/2)$, with the second derivative $I_{\lambda\lambda}(1,1/2) = 1/2$. Thus, by a Taylor expansion to second order of $I(\lambda,p)$ at $(1,1/2)$, the estimates (7.9) and (7.10) result with

$$|I(\frac{m}{\ell},p_k) - \frac{1}{k^2}| \leq \frac{C_2}{k^2\log k} \tag{7.12}$$

for some $C_2 < \infty$, all k large enough and m,ℓ in the range considered here. Since $|\ell - \zeta k^2 \log k| \leq 2k$, combining (7.11) and (7.12) we establish (7.8). □

In Section 8 we control the second moment of the n-perfectness property. To do this, we need to consider excursions between disks centered at $x \in S$ as well as those between disks centered at $y \in S$, $y \neq x$. The radial symmetry we used in proving Lemma 7.1 is hence lost. The next Lemma shows that, in terms of the number of excursions, not much is lost when we condition on a certain σ-algebra $\mathcal{G}_l^y$ which contains more information than just the number of excursions in the previous level. To define $\mathcal{G}_l^y$, let $\tau_0 = 0$ and for $i = 1,2,\dots$ let

$$\begin{aligned}
\tau_{2i-1} &= \inf\{t \geq \tau_{2i-2} :\ w_t \in \partial D(y,\epsilon_l)\} \\
\tau_{2i} &= \inf\{t \geq \tau_{2i-1} :\ w_t \in \partial D(y,\epsilon_{l-1})\}.
\end{aligned}$$

Thus, $N_l^y(1) = \max\{i\ :\ \tau_{2i-1} < \sigma_{y,1}\}$. For each $j = 1,2,\dots,N_l^y(1)$ let

$$e^{(j)} = \{w_{\tau_{2j-2}+t} : 0 \leq t \leq \tau_{2j-1} - \tau_{2j-2}\}$$

be the j'th excursion from $\partial D(y,\epsilon_{l-1})$ to $\partial D(y,\epsilon_l)$ (when $j=1$ the excursion begins at the origin). Finally, let

$$e^{(N_l^y(1)+1)} = \{w_{\tau_{2N_l^y(1)}+t} : 0 \leq t \leq \sigma_{y,1} - \tau_{2N_l^y(1)}\}.$$

We let $J_l := \{l+1,\dots,n\}$ and take $\mathcal{G}_l^y$ to be the σ-algebra generated by the excursions $e^{(1)},\dots,e^{(N_l^y(1))}$, $e^{(N_l^y(1)+1)}$.

Lemma 7.3. *There exists $C_0 < \infty$ such that for any $2 \leq l \leq n-1$, $|m_l - n_l| \leq l$ and all $y \in S$,*

$$\begin{aligned}
&\mathbf{P}(N_k^y(1) = m_k; k \in J_l \,|\, N_l^y(1) = m_l, \mathcal{G}_l^y) \\
&\leq C_0 \prod_{k=l}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,|\, L_k = m_k\right)\ . \qquad (7.13)
\end{aligned}$$

The key to the proof of Lemma 7.3 is to demonstrate that the number of Brownian excursions involving concentric disks of radii ϵ_k, $k \in J_l$ prior to first exiting the disk of radius ϵ_{l-1} is almost independent of the initial and final points of the overall excursion between the ϵ_l and ϵ_{l-1} disks. The next lemma, proven in Section 9, provides uniform estimates sufficient for this task.

Lemma 7.4. *For $l \geq 2$ and a Brownian path starting at $z \in \partial D(y, \epsilon_l)$, let Z_k, $k \in J_l$, denote the number of excursions of the path from $\partial D(y, \epsilon_{k-1})$ to $\partial D(y, \epsilon_k)$, prior to $\bar\tau = \inf\{t > 0 : w_t \in \partial D(y, \epsilon_{l-1})\}$. Then, for some $c < \infty$ and all $\{m_k : k \in J_l\}$, uniformly in $v \in \partial D(y, \epsilon_{l-1})$ and y,*

(7.14)
$$\mathbf{P}^z(Z_k = m_k,\, k \in J_l \,\big|\, w_{\bar\tau} = v) \leq (1 + cl^{-3})\mathbf{P}^z(Z_k = m_k,\, k \in J_l)\ .$$

Proof of Lemma 7.3: Fixing $l \geq 2$ and $y \in S$, let $Z_k^{(j)}$, $k \in J_l$ denote the number of excursions from $\partial D(y, \epsilon_{k-1})$ to $\partial D(y, \epsilon_k)$ during the j-th excursion that the Brownian path w_t makes from $\partial D(y, \epsilon_l)$ to $\partial D(y, \epsilon_{l-1})$. The lemma trivially applies when $m_l = 0$. Considering hereafter $m_l > 0$, since $0 \notin D(y, \epsilon_1)$ we have that conditioned upon $\{N_l^y(1) = m_l\}$,

$$N_k^y(1) = \sum_{j=1}^{m_l} Z_k^{(j)} \qquad k \in J_l\ . \tag{7.15}$$

Conditioned upon $\mathcal{G}_l^y$, the random vectors $\{Z_k^{(j)}, k \in J_l\}$ are independent for $j = 1, 2, \dots, m_l$. Moreover, $\{Z_k^{(j)}, k \in J_l\}$ then has the conditional law of $\{Z_k, k \in J_l\}$ of Lemma 7.4 for some random $z_j \in \partial D(y, \epsilon_l)$ and $v_j \in \partial D(y, \epsilon_{l-1})$, both measurable on $\mathcal{G}_l^y$ (as z_j is the final point of $e^{(j)}$, the j-th excursion from $\partial D(y, \epsilon_{l-1})$ to $\partial D(y, \epsilon_l)$ and v_j is the initial point of the $(j+1)$-st such excursion $e^{(j+1)}$). Hence,

$$\begin{aligned}
&\mathbf{P}(N_k^y(1) = m_k,\, k \in J_l \big|\, N_l^y(1) = m_l, \mathcal{G}_l^y) \\
&= \sum_{\mathcal{P}_l} \prod_{j=1}^{m_l} \mathbf{P}^{z_j}(Z_k^{(j)} = m_k^{(j)},\, k \in J_l \,\big|\, w_{\bar\tau^{(j)}} = v_j)\,,
\end{aligned}$$

where the sum runs over the set $\mathcal{P}_l$ of all partitions $m_k = \sum_{j=1}^{m_l} m_k^{(j)}$, $k \in J_l$. By the uniform upper bound of (7.14) this is

$$\begin{aligned}
&\leq \sum_{\mathcal{P}_l} \prod_{j=1}^{m_l} (1 + cl^{-3})\mathbf{P}^{z_j}(Z_k^{(j)} = m_k^{(j)},\, k \in J_l) \\
&= (1 + cl^{-3})^{m_l} \mathbf{P}\left(N_k^y(1) = m_k,\, k \in J_l \big|\, N_l^y(1) = m_l\right)\ .
\end{aligned}$$

Since $m_l = O(l^2 \log l)$ we thus get the bound (7.13) by the representation used in the proof of Lemma 7.1. □

8. Second moment estimates

Recall that $N_k^x(\rho)$ for $x \in S$, $k \geq 2$, $\rho > \epsilon_1$ denotes the number of excursions from $\partial D(x, \epsilon_{k-1})$ to $\partial D(x, \epsilon_k)$ prior to $\sigma_{x,\rho}$ and as such $N_k^x(1/2) \leq N_k^x(1) \leq N_k^x(2)$. With $n_k = \zeta k^2 \log k$ we shall write $N \overset{k}{\sim} n_k$ if $|N - n_k| \leq k$. Relying upon the first moment estimates of Lemmas 7.1 and 7.3, we next obtain an upper bound related to the second moment of the n-perfectness property. In particular, the inequality (3.12) is a direct consequence of this bound and (7.2).

Lemma 8.1. *For q_n of (7.1), some $\gamma_l \to 0$ and all $x, y \in S$ such that $|x-y| \geq 2\epsilon_l$*

$$\mathbf{P}\left(x \text{ and } y \text{ are n-perfect }\right) \leq q_n^2\, (l!)^{\zeta+\gamma_l}\,. \tag{8.1}$$

Proof of Lemma 8.1: Necessarily $l \geq 2$. We may and shall assume without loss of generality that n is large enough for $n_{n-2} \geq n-1$. Furthermore, assuming without loss of generality that $f(n) := ((n-1)!)^{\zeta+\gamma_{n-1}} \geq (n!)^{\zeta+\delta_n}$ is nondecreasing, it suffices by (7.1) to consider only $l \leq n-2$. Similarly, fixing $x, y \in S$, we may and shall take the minimal value of l such that $|x-y| \geq 2\epsilon_l$. In this case, $D(y,\epsilon_l) \cap \partial D(x,\epsilon_k) = \emptyset$ for all $k \neq l-1$. Moreover, $0 \in D(x,1/2) \subset D(y,1)$ implying that $\sigma_{x,1/2} \leq \sigma_{y,1}$, and, recalling the σ-algebras $\mathcal{G}^y_{l+1}$ defined above Lemma 7.3, $\{N^x_k(1/2) \overset{k}{\sim} n_k\}$ are measurable on $\mathcal{G}^y_{l+1}$ for all $k \neq l-1, l$. With $J_l := \{l+1, \dots, n\}$ and $I_l := \{2, \dots, l-2, l+1, \dots, n\}$, we note that

$$\{x, y \text{ are n-perfect}\} \subset \{N^x_k(1/2) \overset{k}{\sim} n_k,\ k \in I_l\} \bigcap \{N^y_k(1) \overset{k}{\sim} n_k,\ k \in J_l\}\,.$$

Let $\Gamma(I_l) := \{m_2, \dots, m_n : |m_k - n_k| \leq k, k \in I_l\}$ and $\Gamma(J_l) := \{m_{l+1}, \dots, m_n : |m_k - n_k| \leq k, k \in J_l\}$. Then, using (7.13) in the second inequality and the representation (7.7) of Lemma 7.1 in the third,

$$\begin{aligned}
&\mathbf{P}\left(x \text{ and } y \text{ are n-perfect }\right)\\
&\quad\leq \sum_{\Gamma(J_l)} \mathbb{E}\big[\mathbf{P}(N^y_k(1) = m_k,\ k \in J_{l+1} \,\big|\, N^y_{l+1}(1) = m_{l+1}, \mathcal{G}^y_{l+1})\ ;\\
&\qquad\qquad\qquad N^x_k(1/2) \overset{k}{\sim} n_k,\ k \in I_l\big]\\
&\quad\leq C_0\Big[\sum_{\Gamma(J_l)} \prod_{k=l+1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right)\Big]\mathbf{P}\left(N^x_k(1/2) \overset{k}{\sim} n_k,\ k \in I_l\right)\\
&\quad\leq C_0\Big[\sum_{\Gamma(J_l)} \prod_{k=l+1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right)\Big]\\
&\qquad\qquad\cdot\Big[\sum_{\Gamma(I_l)} \prod_{k=1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right)\Big]\\
&\quad\leq C_0\, q_{l-2}\,\Big[\sum_{\Gamma(J_l)} \prod_{k=l+1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right)\Big]^2
\end{aligned} \tag{8.2}$$

(where $m_1 = 1$, $q_0 = q_1 := 1$). By (7.7) and the bounds of Lemma 7.2 we have the inequality,

$$
\begin{aligned}
q_n &= \sum_{\substack{m_2,\dots,m_n \\ |m_k - n_k| \le k}} \prod_{k=1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right) \\
&\ge q_{l+1} \inf_{|m_{l+1} - n_{l+1}| \le l+1} \sum_{\Gamma(J_{l+1})} \prod_{k=l+1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right) \\
&\ge q_{l+1} C^{-2} \sup_{|m_{l+1} - n_{l+1}| \le l+1} \sum_{\Gamma(J_{l+1})} \prod_{k=l+1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right) \\
&\ge q_{l+1} C^{-2} (2l+3)^{-1} \sum_{\Gamma(J_l)} \prod_{k=l+1}^{n-1} \mathbf{P}\left(L_{k+1} = m_{k+1} \,\middle|\, L_k = m_k\right) .
\end{aligned}
\tag{8.3}
$$

Combining (8.2) and (8.3), we see that

$$\mathbf{P}\left(x \text{ and } y \text{ are n-perfect}\right) \le C_0 q_{l-2} \Big[\frac{q_n C^2 (2l+3)}{q_{l+1}} \Big]^2 ,$$

and (8.1) follows from the estimate of (7.1). □

9. Conditional excursion probabilities

Proof of Lemma 7.4: Without loss of generality we can take $y = 0$. Fixing $l \ge 2$ and $z \in \partial D(0, \epsilon_l)$ it suffices to consider $\{m_k,\, k \in J_l\}$ for which $\mathbf{P}^z\left(Z_k = m_k,\, k \in J_l\right) > 0$. Fix such $\{m_k,\, k \in J_l\}$ and a positive continuous function g on $\partial D(0, \epsilon_{l-1})$. Let $\bar\tau = \inf\{t : w_t \in \partial D(0, \epsilon_{l-1})\}$, $\tau_0 = 0$ and for $i = 0, 1, \dots$ define

$$
\begin{aligned}
\tau_{2i+1} &= \inf\{t \ge \tau_{2i} : w_t \in \partial D(0, \epsilon_{l+1}) \cup \partial D(0, \epsilon_{l-1})\} \\
\tau_{2i+2} &= \inf\{t \ge \tau_{2i+1} : w_t \in \partial D(0, \epsilon_l)\} .
\end{aligned}
$$

Set $j = m_{l+1}$ and let Z_k^j, $k \in J_l$ be the corresponding number of excursions by the Brownian path prior to time τ_{2j}. Then, by the strong Markov property at τ_{2j},

$$
\begin{aligned}
&\mathbb{E}^z \left[g(w_{\bar\tau}); Z_k = m_k,\, k \in J_l \right] \\
&= \mathbb{E}^z \left[\mathbb{E}^{w_{\tau_{2j}}} \left(g(w_{\bar\tau}),\, Z_{l+1} = 0 \right); Z_k^j = m_k,\, k \in J_l, \bar\tau \ge \tau_{2j} \right] ,
\end{aligned}
$$

and

$$\mathbf{P}^z\left(Z_k = m_k,\, k \in J_l\right) = \mathbb{E}^z \left[\mathbb{E}^{w_{\tau_{2j}}} \left(Z_{l+1} = 0 \right); Z_k^j = m_k,\, k \in J_l, \bar\tau \ge \tau_{2j} \right] .$$

Consequently,

$$\begin{aligned}&\mathbb{E}^z[g(w_{\bar\tau}); Z_k = m_k,\, k\in J_l]\\ &\le \mathbf{P}^z(Z_k = m_k,\, k\in J_l)\sup_{|x|=\epsilon_l}\frac{\mathbb{E}^x(g(w_{\bar\tau});\, Z_{l+1}=0)}{\mathbb{E}^x(Z_{l+1}=0)},\end{aligned}$$

and, using again the strong Markov property at time τ_2,

$$\begin{aligned}\mathbb{E}^x(g(w_{\bar\tau});\, Z_{l+1}=0) &= \mathbb{E}^x(g(w_{\bar\tau})) - \mathbb{E}^x(\mathbb{E}^{w_{\tau_2}}(g(w_{\bar\tau}));\, Z_{l+1}\ge 1)\\ &\le \mathbb{E}^x(g(w_{\bar\tau})) - \mathbb{E}^x(Z_{l+1}\ge 1)\inf_{|y|=\epsilon_l}\mathbb{E}^y(g(w_{\bar\tau})).\end{aligned}$$

Since $\mathbb{E}^x(Z_{l+1}\ge 1) = p_l$ whenever $|x| = \epsilon_l$, c.f. (7.3), it thus follows that

$$\begin{aligned}&\mathbb{E}^z[g(w_{\bar\tau}); Z_k = m_k,\, k\in J_l]\\ \le\ &\mathbf{P}^z(Z_k = m_k,\, k\in J_l)\\ &\cdot\mathbb{E}^z(g(w_{\bar\tau}))(1-p_l)^{-1}\left\{\frac{\sup_{|x|=\epsilon_l}E^x(g(w_{\bar\tau}))}{\inf_{|y|=\epsilon_l}E^y(g(w_{\bar\tau}))} - p_l\right\}.\end{aligned} \tag{9.1}$$

Recall that

$$\mathbb{E}^{z'}g(w_{\bar\tau}) = \int_{\partial D(0,\epsilon_{l-1})} g(u)K_{\epsilon_{l-1}}(u,z')du,$$

where $K_r(u,z') = \frac{r^2-|z'|^2}{|u-z'|^2}$ is the *Poisson kernel*. Therefore, we get the Harnack inequality

$$\frac{\sup_{|x|=\epsilon_l}E^x(g(w_{\bar\tau}))}{\inf_{|y|=\epsilon_l}E^y(g(w_{\bar\tau}))} \le \frac{\max_{|x|=\epsilon_l,|u|=\epsilon_{l-1}}K_{\epsilon_{l-1}}(u,x)}{\min_{|y|=\epsilon_l,|u|=\epsilon_{l-1}}K_{\epsilon_{l-1}}(u,y)} = \frac{(\epsilon_{l-1}+\epsilon_l)^2}{(\epsilon_{l-1}-\epsilon_l)^2}. \tag{9.2}$$

With $\epsilon_{l-1} = l^3\epsilon_l$, we get from (9.1) and (9.2) that for some universal constant $c<\infty$,

$$\mathbb{E}^z[g(w_{\bar\tau}); Z_k = m_k,\, k\in J_l] \le (1+cl^{-3})\mathbf{P}^z(Z_k = m_k,\, k\in J_l)\,E^z(g(w_{\bar\tau}))$$

and since this bound is independent of g we obtain (7.14). □

10. Complements and unsolved problems

1. In [3] and the present paper, we analyzed Brownian occupation measure where it is exceptionally 'thick'. To describe completely the multifractal structure of the measure, an analysis of 'thin points' is needed. In [4] we show that

$$\lim_{\varepsilon\to 0}\inf_{t\in[0,1]}\frac{\mu_1^w(D(w_t,\varepsilon))}{\varepsilon^2/\log\frac{1}{\varepsilon}} = 1, \qquad a.s. \tag{10.1}$$

We also show that for any $a>1$,

$$\dim\{x\in D(0,1) :\ \liminf_{\varepsilon\to 0}\frac{\mu_1^w(D(x,\varepsilon))}{\varepsilon^2/\log\frac{1}{\varepsilon}} = a\} = 2-2/a \qquad a.s. \tag{10.2}$$

We call a point $x\in D(0,1)$ on the Brownian path a *thin point* if x is in the set considered in (10.1) for some $a>0$. In contrast to the situation for thick points, the results (10.1) and (10.2) for thin points hold for all dimensions $d \geq 2$.

2. The "average" occupation measure of small balls by planar Brownian motion was recently investigated by P. Mörters [14]. He showed that μ_1^w has an *average density of order three* with respect to the gauge function $\varphi(\varepsilon) = \varepsilon^2 \cdot \log \frac{1}{\varepsilon}$, *i.e.*, almost surely,

$$\lim_{r \downarrow 0} \frac{1}{\log|\log r|} \int_r^{1/e} \frac{\mu_1^w(D(x,\varepsilon))}{\varphi(\varepsilon)} \frac{d\varepsilon}{|\varepsilon \log \varepsilon|} = 2 \quad \text{at } \mu_1^w\text{-almost every } x.$$

3. Computation of Laplace transforms is a traditional component of multifractal analysis, and in our work on transient Brownian motion [3] Laplace transforms (exponential moments of occupation measure) were used to determine the coarse multifractal spectrum. In the present paper we obtained the coarse spectrum directly, as this was easier than computation of exponential moments. We believe that

$$\lim_{\epsilon \to 0} \int_0^1 \exp\left(\lambda \mu_1^w\left(D(W_t, \epsilon)\right) / (\epsilon^2 \log 1/\epsilon)\right) dt = \left(\frac{1}{1-\lambda}\right)^2 \quad a.s.\ \forall \lambda < 1. \tag{10.3}$$

Presently, we can only show this for $\lambda < 1/2$. The general result would follow by analyticity arguments if one could prove that

$$\limsup_{\epsilon \to 0} \int_0^1 \exp\left(\lambda \mu_1^w\left(D(W_t, \epsilon)\right) / (\epsilon^2 \log 1/\epsilon)\right) dt < \infty \ a.s.\ \forall \lambda < 1\,; \tag{10.4}$$

this 'almost' follows from Theorem 1.4.

4. Next, we discuss briefly the packing dimension analogue of Theorem 1.3; consult Mattila (1995) for background on packing dimension, Minkowski dimension and their relation. The set of consistently thick points $\mathsf{CThick}_{\geq a}$, defined in (4.2), has different packing dimension from the set $\mathsf{Thick}_{\geq a}$, defined in (2.6). Namely, for every $a \in (0, 2]$,

$$\dim_P(\mathsf{CThick}_{\geq a}) = 2 - a\,, \tag{10.5}$$

$$\dim_P(\mathsf{Thick}_{\geq a}) = 2\,. \tag{10.6}$$

To justify (10.5), we use the notation of Section 2. The sets $\mathcal{A}_n$, defined in (2.11), satisfy

$$|\mathcal{A}_n| \leq (\tilde{\epsilon}_n)^{(1-11\delta)a-2} \tag{10.7}$$

for all large n, by (2.13) and Borel-Cantelli.

Recall the discs $\mathcal{V}_{n,j} = D(x_j, \delta\tilde{\epsilon}_n)$ defined after (2.13), and denote $\mathcal{V}_n = \cup_{j \in \mathcal{A}_n} \mathcal{V}_{n,j}$. By (10.7), the upper Minkowski dimension of $\mathcal{V}_\ell^* = \cap_{n \geq \ell} \mathcal{V}_n$ is at most $2 - (1-11\delta)a$. It is easy to see that $\mathsf{CThick}_{\geq a} \subset \cup_{\ell \geq 1} \mathcal{V}_\ell^*$, whence $\dim_P(\mathsf{CThick}_{\geq a}) \leq 2 - (1-11\delta)a$. Since δ can be taken arbitrarily small, while $\dim_P(\mathsf{CThick}_{\geq a}) \geq \dim(\mathsf{CThick}_{\geq a})$, this proves (10.5).

To prove (10.6), it clearly suffices to consider $a = 2$. Recall that $\bar\theta = \inf\{t : |w_t| = 1\}$. For each $n \geq 1$, let

$$V_n := \bigcup_{0<\varepsilon<1/n} \Big\{0 < t < \bar\theta : \frac{\mu^w_{\bar\theta}(D(w_t,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} > 2 - 1/n\Big\}.$$

It is easy to check, e.g. by applying Theorem 1.2 with an arbitrary T replacing $\bar\theta$ there, and using the shift invariance of Brownian motion, that for any $n \geq 1$, almost surely V_n is an open dense set in $(0, \bar\theta)$; by [3, Cor. 2.4], $\dim_P(\cap_n V_n) = 1$ a.s. The set

$$\Big\{0 \leq t \leq \bar\theta : \limsup_{\varepsilon\to 0} \frac{\mu^w_{\bar\theta}(D(w_t,\varepsilon))}{\varepsilon^2 \left(\log \frac{1}{\varepsilon}\right)^2} \geq 2\Big\}, \tag{10.8}$$

contains $\cap_n V_n$, so it has packing dimension 1. Finally, $\mathsf{Thick}_{\geq 2}$ is the image under planar Brownian motion of the set in (10.8); hence the uniform doubling of packing dimension by planar Brownian motion, see [17, Cor. 5.8], yields (10.6).

5. In Theorem 5.1 we assumed that the random walk increments have finite moments of all orders. We suspect that finite second moments suffice, but our method only gives the following result:

   Under the assumptions of Theorem 5.1, except that the moment assumption on the increments is relaxed to $\mathbb{E}|X|^m < \infty$ for some real $m \geq 2$, we have

$$\frac{1-1/m}{\pi_\Gamma} \leq \liminf_{n\to\infty} \frac{T^*_n}{(\log n)^2} \leq \limsup_{n\to\infty} \frac{T^*_n}{(\log n)^2} \leq \frac{1}{\pi_\Gamma} \quad a.s. \tag{10.9}$$

6. In this paper we focussed on Brownian occupation measure in dimension two, the critical dimension for recurrence. Other natural random measures for which we expect analogous results are the occupation measure of the symmetric Cauchy process on the line, and the projected intersection local times for several planar Brownian motions. Establishing such results is a challenging problem.

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