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RISK AVERSION AND THE MARTINGALE PROPERTY OF STOCK PRICES*

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1. INTRODUCTION AND SUMMARY OF CONCLUSIONS

RECENT EMPIRICAL STUDIES of the random properties of stock prices[1] have supported the conclusion that rates of return on stock follow a martingale—i.e., that the expected rate of return on stock conditional on past realized rates of return is always equal to its unconditional expectation. In addition, the martingale property has received theoretical support from recent work by Samuelson [10].[2] However, Samuelson's result depends on the assumption that investors require an exogenously given expected rate of return. It is natural to inquire whether the martingale property can be derived when the assumption of a given expected rate of return is relaxed. That question will be discussed in this paper.

If it is no longer assumed that the expected rate of return may be taken as given, then it becomes necessary to consider how the expected rate of return is determined, and this involves analyzing the relation between the riskiness of stock and the risk-aversion of investors. We are led to consider models of portfolio selection of the type developed by Tobin [13], [14] and Markowitz [6], and the associated models of capital market equilibrium of Sharpe [12] and Lintner [5], since these deal explicitly with this question. However, it is apparent that models of the Sharpe-Lintner type, though they do relate the expected rate of return to the optimizing behavior of risk-averse investors, can cast no light on the martingale question. This is so because these models assume a one-period framework, with the expected value and variance of the next-period price being taken as given, and therefore cannot generate an intertemporal probability distribution.

It is necessary somehow to dynamize the Sharpe-Lintner model; i.e., to modify the model so that it generates an intertemporal distribution of asset prices. In order to do this it is assumed that investors bid for a financial asset ("stock") which constitutes a claim to a random intertemporal stream of earnings with known probability distribution, rather than for a security which matures in the next period as in the Sharpe-Lintner one-period portfolio models. It is assumed that investors have a choice between risky stock and a riskless asset earning a constant exogenous rate of return, as in the one-period portfolio models.

In Section 2 such a model is developed, though only for a restricted class of

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[1] Cf., for example, Fama [2], Granger and Morgenstern [3].

[2] Samuelson's model derives the martingale property for futures prices rather than for an equity asset. However, since a share of stock can be regarded as a set of futures claims due to mature at successive intervals, his proof evidently carries over. For a formal demonstration of the result for stock prices. Cf. [4].

earnings distributions. It is seen that the analysis of the multiperiod capital market equilibrium problem is distinguished from that of the one-period model in at least two major respects:

(1) It is necessary to give careful consideration to the problem of the formation of expectations in the dynamic version. In a single-period model it is certainly reasonable to assume that subjective expectations of future asset prices are both given and unbiased, but in a multiperiod model the problem is more difficult. Evidently (conditional) expectations will depend on the realization of the system's past and present random elements, the distribution of which will be endogenous and dependent, in turn, on expectations.[3] In such a context it is no longer possible to take expectations as given if the reasonable assumption that these expectations are unbiased is to be maintained.

(2) As a result of the multiperiod character of the model, the solution that is sought takes the form, not of a level for asset prices, but of a function relating asset prices to the past realization of earnings. The form of the asset price function is derived (as is the asset price level in the single-period case) by exploiting the requirement that markets be cleared, but with this difference: in the dynamic case it is required that markets be cleared for any realization of past earnings. It is seen that under the conditions of the model this imposes restrictions sufficient to determine the form of the price function.

Since the model generates an intertemporal probability distribution for rates of return on stock, it is possible in Section 3 to return to our initial question: does the derived distribution satisfy a martingale property? On intuitive grounds we would expect the answer to be negative since the martingale property relates only to the first moment of a distribution, while risk-aversion involves (in our specification) a trade off between the first and second moments. It is seen that this conclusion is correct.

The formal derivation of the probability distribution of rates of return is valid only under the extremely restrictive assumptions presented in Section 2. It is natural to inquire whether anything can be said about the distribution of rates of return under more general conditions. In Section 4 it is shown that in the general case successive rates of return may be either positively or negatively correlated, depending on how the expectation and variance of the next-period return depend on past realized earnings. In the absence of prior information about the sign of the correlation coefficient relating successive rates of return, it is clearly reasonable to conclude that the empirical evidence in favor of the martingale is at least not inconsistent with theory.

In this paper it is demonstrated, then, not that any particular systematic departure from the martingale property is to be expected, but only that under risk-aversion no rigorous theoretical justification for an exact martingale property is available. It follows that the notion of "efficient" or "well-functioning" capital markets cannot under general conditions be associated with a particular class of probability distributions, except as an approximation.

[3] For a brief discussion of this problem in a somewhat different context, cf. Diamond [1].

2. DEVELOPMENT OF THE MODEL

In order to adapt the standard portfolio model to a dynamic setting, several assumptions about investor behavior and the available assets are needed:

ASSUMPTION 1. Two assets are available: (1) a risk-free security which earns a constant exogenous rate of return $r^*(r^* > 0)$, and which can be either held long or sold short, and (2) homogeneous "stock," which is title to a random stream of earnings, the distribution of which is stationary and known. All earnings on stock are paid out in dividends, and all the stock must be held by the investors. There are no new issues, so the price of the equity is equal to the total value of the stock. Security markets are competitive.

ASSUMPTION 2. There are n investors, each possessing as initial wealth an equal share of the equity and an equal allocation of cash c. The investors' risk preferences are identical, and can be represented by a utility function in the mean and variance of next-period wealth. Each investor's utility function $U[E(\tilde{w}^i_{t+1}), V(\tilde{w}^i_{t+1})]$ satisfies $U_1 > 0$, $U_2 < 0$, with the matrix

$$\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

strictly positive definite (i.e., if the indifference contours are plotted with standard deviation on the abscissa and expected wealth on the ordinate, these curves are concave upward). Also, investors exhibit constant absolute risk-aversion (U is such that U_2/U_1 is not a function of $E[\tilde{w}^i_{t+1}]$).[4]

Here E and V represent the expectations and variance operators, w^i_t denotes the i-th investor's wealth at time t, the tilde signifies a random variable and the subscripts of U indicate partial derivatives.

The assumption that each investor's initial wealth consists partly of equity embodies an essential characteristic of financial models incorporating assets with a multiperiod or indefinite maturity: the interdependence between investors' collective demand behavior and each investor's initial wealth. Evidently the reason for the interdependence is that the asset which is being priced in the current period is the same asset as that which constitutes investors' initial wealth. Since in the real world stocks and bonds make up a major proportion of most investors' portfolios, rather than assets similar to Treasury bills as assumed in

[4] The restriction that U_2/U_1 is not a function of expected wealth is identified with constant absolute risk-aversion because it is the analogue in the mean-variance framework of the requirement that $-U''/U'$ is not a function of wealth under the expected utility approach (Pratt [9]). To see this, recall that the defining property of constant absolute risk aversion is that the risk premium on a given risk is not a function of wealth. The analogous property in the mean-variance framework obtains if U_2/U_1 is not a function of expected wealth. To show this, it is noted first that the risk premium π on a risky asset with variance y may be defined by

$$U(x+\pi(x, y), y) = U(x, 0)\,.$$

By differentiating with respect to y, it is evident that $d\pi/dy$, and hence π, is independent of x if and only if U_2/U_1 is.

many theoretical models, the stipulated interdependence is a valuable property of the model.

The assumption of constant absolute risk-aversion requires some motivation. It will be seen below that a necessary condition for the solution of the model is that the functional relationship between earnings and stock prices remain stable over time. This, in turn, will require that the ratio U_2/U_1 remain constant over time, since U_2/U_1 is one of the determinants of the form of the function linking earnings and prices. But for a general utility function, U_2/U_1 will depend on the level of expected next-period wealth, which will fluctuate over time as stock prices rise and fall. The assumption of constant absolute risk-aversion will prove necessary to guarantee the constancy of U_2/U_1.[5]

ASSUMPTION 3. Each investor can always make an unbiased forecast of the price of stock in the next period.

We begin the analysis by considering the i-th investor's choice of portfolio. His problem is to select the optimal level of h_t^i, the proportion of the total equity he will hold; the remainder of his wealth will be invested in the risk-free asset. First the relations between h_t^i and the mean and variance of next-period wealth are developed. The expected value of next-period wealth will equal the sum of the next-period value of the investor's holdings of the risk-free asset, the expected value of earnings in the next period on the stock he elects to hold, and the expected value of the stock itself.

The next-period value of the investor's holdings of the risk-free asset is the product of the amount invested in the risk-free asset, equal to initial wealth $(c + p_t/n)$ minus the value of stock purchased $(p_t h_t^i)$, and the constant $1 + r^*$. The expected value of the i-th investor's earnings and stock holdings in the next period equals $h_t^i(x_{t+1}^e + p_{t+1}^e)$, where x_t and p_t are the earnings on and price of all the stock at time t, and where superscript "e" denotes an expectation conditioned by the variates $x_t, x_{t-1}, \ldots$ (rather than an unconditional expectation). Assumptions 1 and 3 guarantee that the investor is able to form expectations of earnings and stock prices in the next period. Summing these elements, the expected next-period wealth is given by

$$(1) \qquad E(\tilde{w}_{t+1}^i; h_t^i) \equiv (c + p_t/n - p_t h_t^i)(1 + r^*) + h_t^i(x_{t+1}^e + p_{t+1}^e), \qquad i = 1, \ldots, n.$$

[5] In view of the multiperiod setting of the model, it might seem natural to integrate the portfolio choice problem with the consumption-investment problem by deriving decision rules for both from dynamic utility maximization along the lines of Samuelson [**11**], Merton [**7**], and Mossin [**8**]. However, the requirement of constant absolute risk-aversion, which would carry over into the more general case, implies that in a dynamic setting the consumption-savings decision and the portfolio decision are made separately (see Samuelson [**11**] for a discussion of this decomposition in the essentially similar case of constant relative risk-aversion). For this reason no true generalization can be achieved without relaxing the assumption of constant absolute risk-aversion, and that would fundamentally alter the model.

The basic conclusion of this paper, that rates of return on stock will follow an exact martingale only as a special case, is obviously not impaired by the lack of generality in the specification of the utility function.

In order to facilitate the solution of the model, it is useful to express expected next-period wealth in terms of the expected excess return on all the stock, defined as the difference between the expected return on the stock in the next period, including earnings, and the return on the investment of an equivalent amount in the risk-free asset. This definition may be expressed by

$$e(x_t, x_{t-1}, \ldots) \equiv x^e_{t+1} + p^e_{t+1} - p_t\,(1 + r^*), \tag{2}$$

where e_t is the expected excess return on all the equity. Substituting (2) into (1) and simplifying, the expression for expected next-period wealth becomes

$$E(\tilde{w}^i_{t+1};\, h^i_t) \equiv (c + p_t/n)(1 + r^*) + h^i_t e(x_t, x_{t-1}, \ldots). \tag{3}$$

The variance of the return on a fraction h^i_t of the total equity is given by

$$\begin{aligned} V(\tilde{w}^i_{t+1};\, h^i_t) &= (h^i_t)^2\{[E(\tilde{x}_{t+1} + \tilde{p}_{t+1})^2 \,|\, x_t, x_{t-1}, \ldots] \\ &-[E(\tilde{x}_{t+1} + \tilde{p}_{t+1} \,|\, x_t, x_{t-1}, \ldots)]^2\} \equiv (h^i_t)^2\, v(x_t, x_{t-1}, \ldots), \end{aligned} \tag{4}$$

where v_t is the variance of the return on all the equity.

The investor must solve the following problem:

$$\max_{h^i_t} U[E(\tilde{w}^i_{t+1};\, h^i_t),\, V(\tilde{w}^i_{t+1};\, h^i_t)],$$

where the expectation and variance of next-period wealth are as given in (3) and (4). The first-order condition is

$$U_1 e(x_t, x_{t-1}, \ldots) + 2U_2 h^i_t v(x_t, x_{t-1}, \ldots) = 0. \tag{5}$$

This optimization, so far identical to that of the static two-asset portfolio theory, is easily represented graphically (Figure 1). By assumption, the investor's indifference curves are upward-sloping and concave upward. Under competition the rate at which the investor can trade the expected value of next-period wealth against its standard deviation is constant, so the uniqueness of the solution for h^i_t is guaranteed.[6] In market-clearing equilibrium all the equity must be held, and since all investors are identical each will hold an equal share of the equity. Substituting $h^i_t = 1/n$, (5) becomes

$$U_1 e(x_t, x_{t-1}, \ldots) + 2U_2 v(x_t, x_{t-1}, \ldots)/n = 0 \text{ for all } x_t, x_{t-1}, \ldots, \tag{6}$$

defining an implicit dependence between the functions e_t and v_t.

It is at this point that the model departs from the Sharpe-Lintner theory. In the work of Sharpe and Lintner the variance of return in the next period is taken as exogenous and (6) is used to determine expected excess return and therefore the current asset price. Here since the current stock price depends on earnings the variance of return will depend on the distribution of earnings and on the relation of stock prices to earnings, hence must be taken as endogenous. To

[6] Without further restrictions there is no guarantee that a value for h^i_t exists which satisfies the first-order conditions and also the constraint $0 \leq h^i_t \leq 1$. This problem will disappear when the market-clearing constraint is used to derive the slope of the market line, thus assuring an internal maximum.

obtain the restriction needed to close the model it is then necessary to relate e_t and v_t to their underlying determinant, the probability distribution of earnings.

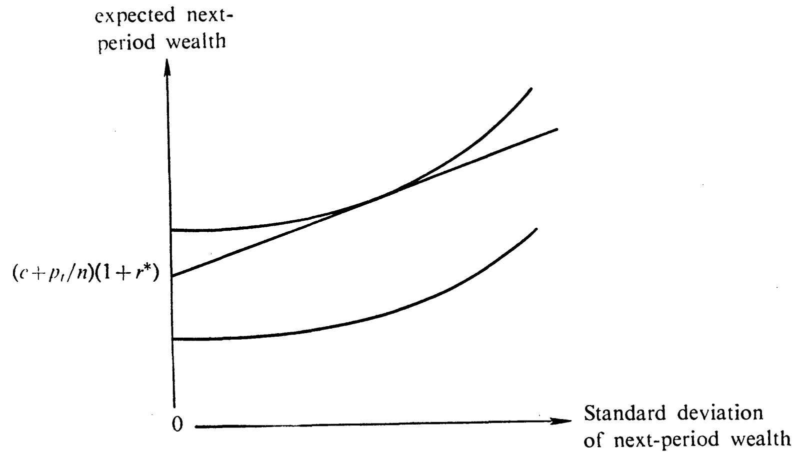

FIGURE 1
OPTIMAL PORTFOLIO CHOICE OF THE INVESTOR

It is not evident, however, exactly how the expected excess return and variance functions are to be related to the distribution of earnings. The form of both functions depends on the probability distribution of next-period stock prices as well as current stock prices. But the probability distribution of stock prices is unknown; indeed, the purpose of this exercise is to derive that distribution. Thus it appears that, paradoxically, it is necessary to know the solution in order to derive it.

As is usually the case, the paradox is more apparent than real. By considering systematically the formal properties any solution to the model must possess, it is possible to determine simultaneously (and in a symmetric fashion) both present stock prices and the probability distribution of next-period stock prices. To see this, it is first noted that, if the model is to be economically meaningful, stock prices in the current period are known. Second, since the probability distribution of earnings is stationary, it may plausibly be assumed that the distribution of stock prices is stationary also. These two conditions imply that there must be some function f relating the current price of stock to current and past earnings (and to all the parameters of the model), and that this function is stationary over time. The function, expressed as

$$p_t = f(x_t, x_{t-1}, \ldots),$$

will be referred to as a *price function* since it relates the price of stock to current and past realizations of earnings.[7] The advantage of this formulation is that it allows a symmetric treatment of the current stock price, which is nonrandom, and the next-period price, which is random. Specifically, the probability distribution of stock prices in the next period, conditional on current and past earnings, may be determined by using the transformation

$$\tilde{p}_{t+1} = f(\tilde{x}_{t+1}, x_t, x_{t-1}, \ldots), \tag{7}$$

where $x_t, x_{t-1}, \ldots$ are taken as real variables, and $\tilde{x}_{t+1}$ is a random variable of which the relevant probability distribution is that conditional on $x_t, x_{t-1}, \ldots$.

The price function formulation may now be used to derive expressions for the expected excess return and variance functions. The term which causes difficulty in the expression for expected excess return, equation (2), is p^e_{t+1}. From Assumption 3, however, it follows that p^e_{t+1}, the subjective stock price expectation in the next period, must coincide with its objective distribution derived from the probability distribution of stock prices as expressed by the price function. Using (7), p^e_{t+1} may be expressed by

$$p^e_{t+1} = \int_{x_{t+1}} f(x_{t+1}, x_t, \ldots) dF(x_{t+1} \mid x_t, x_{t-1}, \ldots).$$

In a similar manner, the expected excess return and variance functions (3) and (4) may be expressed in terms of f by

$$e_t = E_{\tilde{x}_{t+1}}[\tilde{x}_{t+1} + f(\tilde{x}_{t+1}, x_t, \ldots)] - f(x_t, x_{t-1}, \ldots)(1 + r^*)$$
$$v_t = E_{\tilde{x}_{t+1}}[\tilde{x}_{t+1} + f(\tilde{x}_{t+1}, x_t, \ldots)]^2 - \{E_{\tilde{x}_{t+1}}[\tilde{x}_{t+1} + f(\tilde{x}_{t+1}, x_t, \ldots)]\}^2$$

where the conditional expectation is taken.

Without restrictions on the distribution of earnings there is no way to find the function f, or even to determine whether it exists. In order to derive an explicit solution, a specific probability distribution for earnings is adopted:

ASSUMPTION 4. The probability distribution of earnings is given by

$$x_t = \lambda x_{t-1} + \mu + \varepsilon_t, \ |\lambda| < 1, \ E(\varepsilon_t) = 0, \ \mu \geq 0,$$
$$E(\varepsilon_t)^2 = \sigma^2, \ \varepsilon_t \text{ serially independent.}^8$$

Adoption of this linear autoregressive distribution drastically reduces the generality of any derived results, but (for $\lambda \neq 0$) it does preserve the dynamic character of the model, and therefore gives some insight into the functioning of a dynamic portfolio model under more general conditions.

Since under the autoregressive restriction all the information about the conditional distribution of x_{t+1} contained in the past history of the system is

[7] I am indebted to a referee for pointing out that f may usefully be regarded as a "reduced-form" equation for stock prices, since its form depends on all the parameters that influence demand for stock, though f itself does not constitute a demand or supply equation.

[8] The stationary-state solution to the model derived below may be generalized to a steady-state solution simply by replacing n by $n_o(1+g)^t$ in Assumption 2 and x_t by $x_t(1+g)^{-t}$ in Assumption 4.

summarized in x_t, the values taken on by $x_{t-1}, x_{t-2}, \ldots$ are irrelevant to the determination of portfolio choice, and hence also to the analysis of stock price determination. For this reason we may write f as a function of the one argument x_t, rather than as a function of the set $x_t, x_{t-1}, \ldots$. By examining the case in which f is linear it is possible to derive a solution.[9]

If $p_t = \alpha + \beta x_t$ the expected excess return and variance may be readily determined. Since $x^e_{t+1} = \lambda x_t + \mu$, we have that

$$\begin{aligned} & x^e_{t+1} + p^e_{t+1} = \alpha + (1 + \beta)(\lambda x_t + \mu) \text{ and} \\ (8) \qquad & e(x_t, x_{t-1}, \ldots) \equiv x^e_{t+1} + p^e_{t+1} - p_t(1 + r^*) \\ & \quad = [(1 + \beta)\lambda - \beta(1 + r^*)]x_t + (1 + \beta)\mu - \alpha r^*. \end{aligned}$$

The conditional variance of return is simply the variance of $\alpha + (1 + \beta)\tilde{x}_{t+1}$ conditional on x_t, or

$$(9) \qquad v(x_t, x_{t-1}, \ldots) = (1 + \beta)^2\sigma^2.$$

The problem now is to find values of α and β such that the market-clearing equation (6) is satisfied, where e_t and v_t are given by (8) and (9). Under the assumption of constant absolute risk aversion (Assumption 2), the constancy over time of total risk in the system (equation 9) implies that U_2/U_1 is also a constant. This implies in turn that (6) will not be satisfied for all values of x_t unless $e(x_t)$ is constant. But this can only be the case if the coefficient of x_t in equation (8) is zero, which means that β must be given by

$$(10) \qquad \beta = \frac{\lambda}{1 + r^* - \lambda}.$$

Using equations (6), (8), and (9), the market-clearing equation becomes

$$(1 + \beta)\mu - \alpha r^* + 2(U_2/U_1)(1 + \beta)^2\sigma^2/n = 0$$

and α may be expressed in terms of the parameters of the model as

$$\alpha = \frac{2\sigma^2\left(\dfrac{1 + r^*}{1 + r^* - \lambda}\right)^2\left(\dfrac{U_2}{U_1 n}\right) + \left(\dfrac{1 + r^*}{1 + r^* - \lambda}\right)\mu}{r^*}.$$ [10]

[9] A more satisfactory procedure would be to prove analytically that f' is not a function of x_t, since that would demonstrate that the form of f derived below is unique among the set of all differentiable functions. Unfortunately, that procedure involves considerable mathematical difficulty.

[10] If a general von Neumann-Morgenstern utility function is substituted for the more restricted function in mean and variance, the expression for β is the same but the solution for α takes the form of an integral equation [**4**, (55–57)]. Since a relatively simple expression for α facilitates the interpretation of the model, the restricted specification was chosen.

The extension to n risky assets, necessary to complete the parallel with the Sharpe-Lintner model, follows easily if earnings are specified to follow a multivariate rather than univariate linear autoregressive distribution. If a vector-matrix notation is adopted, the solution for α and β are virtually identical to those derived here, and it can be shown that all the Sharpe-Lintner propositions are true in the present model [**4**, (57–58)].

3. RANDOM PROPERTIES OF THE MODEL

A time series of asset prices will be said to follow a martingale if the associated rate of return series satisfies the restriction

$$E(\tilde{r}_t \mid r_{t-1}, r_{t-2}, \ldots) = E(\tilde{r}_t) \text{ for all } r_{t-1}, r_{t-2}, \ldots. \tag{11}$$

In the preceding section an equation relating the price of equity to its current earnings has been derived.[11] By implication the probability distribution of stock prices is determined as well, and also that of rates of return on stock. Does the latter distribution conform to the martingale restriction (11)? We may verify that it does not; it is sufficient to note the identity

$$r_t^e \equiv r^* + e_t/p_t \tag{12}$$

and to recall that expected excess return is constant over time. Current equity prices are the resultant of all past rates of return (assuming a nonstochastic starting point), so it is evident that for $e > 0$ the set of past rates of return does affect the expected value of the current rate of return. Expected excess return will be equal to zero if and only if investors are risk-neutral (or if there is no risk), in which case the martingale property will be satisfied.[12]

It is obvious why the martingale property is violated. Since the variance of total return is constant over time, the variance of return per dollar invested in stock must be related to the current stock price, hence is autocorrelated. It is not surprising to find that when dollar risk is high, investors demand a high expected rate of return, and conversely. It may be concluded that the failure of the probability distribution to conform to the martingale class does not imply any inefficiency in the capital market.

4. GENERALIZATIONS

The result that under risk-aversion rates of return on stock will not satisfy the martingale property was proved only in a very restricted context. The assumptions that investors exhibit constant absolute risk-aversion and that earnings on stock conform to a first-order autoregressive process are particularly unrealistic. Unfortunately, it is difficult to arrive at explicit solutions for the price function, and therefore to derive the distribution of rates of return, when these restrictions are relaxed. However, through examination of the present model it is possible to isolate the condition that must be satisfied if rates of return are to conform to the martingale restriction under any particular utility function and earnings distribution. Except under very special circumstances this condition

[11] A general discussion of the properties of the model presented in the preceding section, though of interest, would be out of place here. For a derivation of the comparative static properties of the model cf. [**4**].

[12] It can be shown that in the model discussed here the value of equity is equal to the discounted value of (expected) future earnings if and only if there is no risk or investors are risk-neutral. This leads us to suspect that there may be a connection between the martingale property and the present-value formula. In [**4**] it is shown that this conjecture is correct.

will not be satisfied, implying that in general the martingale restriction will be violated.

For any given utility function and distribution of earnings there will be an associated price function relating stock prices to current and past earnings. This price function in turn will determine an expected excess return function and a variance function. These functions will exhibit a dependence since they must satisfy a market-clearing equation similar to (6) for any values of their arguments; the form of the dependence is such that each investor will opt to hold exactly $1/n$-th of the equity for any realization of earnings. Now, by rewriting (12) in the form

$$r^e \equiv r^* + \frac{e(x_t, x_{t-1}, \ldots)}{f(x_t, x_{t-1}, \ldots)}$$

it is seen that the martingale restriction will be satisfied only if the form of the dependence between the expected excess return and price functions implied by the earnings distribution is such that the ratio of expected excess return to price remains constant. Under an unrestricted distribution of earnings this simply will not occur, so it may be concluded that in general a dependence will exist between past earnings (hence past rates of return) and the current expected rate of return.[13]

It has been seen in this paper that when the expected rate of return on stock is explained in terms of the portfolio optimization of risk-averse investors rather than simply taken as given, the martingale property fails. How important is this empirically? First, it is likely that changes in the expected rate of return due to changes in estimates of risk are small in comparison with the short-run fluctuations in realized rates of return. Second, we have no knowledge of whether expected excess return will vary more than or less than proportionately with price; equivalently, there is no reason to suspect that successive rates of return will be negatively (as in the restricted case discussed above) rather than positively correlated, or vice versa.

In view of these observations we are led to expect on prior grounds that if capital markets are efficient, rates of return will follow a martingale distribution as a fair approximation even in the presence of risk-aversion. This expectation, plus the results from many empirical studies in support of the same conclusion, implies that for most purposes the simple assumption of the martingale property is acceptable on both theoretical and empirical grounds. Our intention here has not been to challenge the prominent position accorded to the martingale class of distributions in discussions of efficient capital markets, but rather to demonstrate that under general conditions (particularly risk-aversion) the martingale property will be satisfied only as an approximation and that no rigorous theoretical justification for it is available.

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[13] Again, it is noted that under risk-neutrality expected excess return will always be equal to zero, so that the expected rate of return will equal r^* and the martingale property will be satisfied even under a general distribution of earnings.

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