THE SUMMATION OF RANDOM CAUSES AS THE SOURCE OF CYCLIC PROCESSES*

By Eugen Slutzky

I. SCOPE OF THE INVESTIGATION

Almost all of the phenomena of economic life, like many other processes, social, meteorological, and others, occur in sequences of rising and falling movements, like waves. Just as waves following each other on the sea do not repeat each other perfectly, so economic cycles never repeat earlier ones exactly either in duration or in amplitude. Nevertheless, in both cases, it is almost always possible to detect, even in the multitude of individual peculiarities of the phenomena, marks of certain approximate uniformities and regularities. The eye of the observer instinctively discovers on waves of a certain order other smaller waves, so that the idea of harmonic analysis, viz., that of the possibility of expressing the irregularities of the form and the spacing of the waves by means of the summation of regular sinusoidal fluctuations, presents itself to the mind almost spontaneously. If the results of the analysis happen sometimes not to be completely satisfactory, the discrepancies usually will be interpreted as casual deviations superposed on the regular waves. If the analyses of the first and of the second halves of a series give considerably divergent results (such as, for example, were found by Schuster while analyzing sunspot periodicity),[1] it is, even then, possible to find the solution without giving up the basic concept. Such a discrepancy may be the result of the interference of certain factors checking the continuous movement of the process and substituting for the former regularity a new one which sometimes may

* Professor Eugen Slutzky's paper of 1927, "The Summation of Random Causes as the Source of Cyclic Processes," *Problems of Economic Conditions*, ed. by The Conjuncture Institute, Moskva (Moscow), Vol. 3, No. 1, 1927, has in a sense become classic in the field of time-series analysis. While it does not give a complete theory of the time shape that is to be expected when a given linear operator is applied to a random (auto-non-correlated) series, it has given us a number of penetrating and suggestive ideas on this question. It has been, and will no doubt continue to be, highly stimulating for further research on this vast and—not least for business-cycle analysis—most important problem. Unfortunately Professor Slutzky's paper so far has been available only in Russian (with a brief English summary). Some years ago Professor Henry Schultz had the original article translated into English by Mr. Eugene Prostov, and suggested that it be published in Econometrica. At the request of the Editor Professor Slutzky has prepared for our Journal a revised English version with which he has incorporated also a number of important results obtained after 1927.—Editor.

[1] Arthur Schuster, "On the Periodicities of Sunspots," *Phil. Trans.*, Series A, Vol. 206, 1906, p. 76.

even happen to be of the same type as the former one. Empirical series are, unfortunately, seldom long enough to enable one definitely to prove or to refute such an hypothesis. Without dwelling on the history of complicated disputes concerning the above-mentioned problem, I will mention only two circumstances as the starting points for the present investigation—one, so to speak, in the field of chance, the other in the field of strict regularity.

One usually takes the analysis of the periodogram of the series as the basis for the discovery of hidden periodicities. Having obtained from the periodogram the values of the squares of the amplitudes of the sinusoids, calculated by the method of least squares for waves of varying length, we ask whether there is a method of determining those waves which do not arise from chance. Schuster apparently has discovered a suitable method;[2] but we must give up his criterion when we remember that among his assumptions is that of independence of the successive observations. As a general rule we find that the terms of an empirical series are not independent but correlated and at times correlated very closely. This circumstance, as is known, may very perceptibly heighten the oscillation of the derived characteristics of the series, and it is quite conceivable that waves satisfying Schuster's criterion would in fact be casual—just simulating the presence of a strict regularity.[3] Thus we are led to our basic problem: is it possible that a definite structure of a connection between random fluctuations could form them into a system of more or less regular waves? Many laws of physics and biology are based on chance, among them such laws as the second law of thermodynamics and Mendel's laws. But heretofore we have known how regularities could be derived from a chaos of disconnected elements because of the very disconnectedness. In our case we wish to consider the rise of regularity from series of chaotically-random elements because of certain connections imposed upon them.

Suppose we are inclined to believe in the reality of the strict periodicity of a business cycle, such, for example, as the eight-year period postulated by Moore.[4] Then we should encounter another difficulty. Wherein lies the source of the regularity? What is the mechanism of

[2] A. Schuster, "On the Investigation of Hidden Periodicities, etc.," *Terrestrial Magnetism*, Vol. 3, 1898.

[3] The further development of Schuster's methods, which we find in his extremely valuable paper, "The Periodogram of the Magnetic Declination as Obtained from the Records of the Greenwich Observatory during the Years 1871–1895," *Trans. of the Cambridge Philos. Soc.*, Vol. 18, 1900, p. 107, seems to overcome this difficulty. Because it is rather unfinished in mathematical respects, however, the influence of this paper seems not to have been comparable to its importance.

[4] H. L. Moore, *Generating Economic Cycles*, New York, 1923.

causality which, decade after decade, reproduces the same sinusoidal wave which rises and falls on the surface of the social ocean with the regularity of day and night. It is natural that even now, as centuries ago, the eyes of the investigators are raised to the celestial luminaries searching in them for an explanation of human affairs. One can dauntlessly admit one's right to make bold hypotheses, but still should not one try to find out other ways?[5] What means of explanation, however, would be left to us if we decided to give up the hypothesis of the superposition of regular waves complicated only by purely random components? The presence of waves of definite orders, the long waves embracing decades, shorter cycles from approximately five to ten years in length, and finally the very short waves, will always remain a fact begging for explanation. The approximate regularity of the periods is sometimes so distinctly apparent that it, also, cannot be passed by without notice. Thus, in short, *the undulatory character of the processes and the approximate regularity of the waves* are the two facts for which we shall try to find a possible source in random causes combining themselves in their common effect.

The method of the work is a combination of induction and deduction. It was possible to investigate by the deductive method only a few aspects of the problem. Generally speaking, the theory of chance waves is almost entirely a matter of the future. For the sake of this future theory one cannot be too lavish with experiments: it is experiment that shows us totally unexpected facts, thus pointing out problems which otherwise would hardly fall within the field of the investigator.[6]

II. COHERENT SERIES OF CONSEQUENCES OF RANDOM CAUSES AND THEIR MODELS

There are two kinds of chance series: (1) those in which the probability of the appearance, in a given place in the series, of a certain value of the variable, depends on previous or subsequent values of the variable, and (2) those in which it does not. In this way we distinguish

[5] A similar viewpoint is found in the remarkable work of G. U. Yule, "Why Do We Sometimes Get Nonsense-Correlations between Time Series?" *Journal of the Royal Statistical Society*, Vol. 89, 1926. This work approaches our theme rather closely.

[6] The following exposition is based on a large amount of calculation. The author expresses special gratitude to his long-time collaborator, E. N. Pomeranzeva-Ilyinskaya and also to O. V. Gordon, N. F. Rein, M. A. Smirnova and E. V. Luneyeva. The calculations were carefully checked, almost all work having been independently performed by two individuals. It is very unlikely that undetected errors are sufficiently significant to affect to any perceptible degree our final conclusions. A few errors, detected in the course of time in Tables I, III, and IX of the original paper, are noted at the end of this paper, and an error in Figure 7, B_4, has been corrected when it was re-drawn.

between *coherent*[7] and *incoherent* (or random) series. The terms of the series of this second kind are not correlated. In series in which there is correlation between terms, one of the most important characteristics is the value of the coefficient of correlation between terms, considered as a function of the distance between the terms correlated. We shall call it the *correlational function* of the corresponding series and shall limit our investigation to those cases in which the distribution of probabilities remains constant. The coefficient of correlation, then, is exclusively determined by the distance between the terms and not by their place in the series. The coefficient of correlation of each member with itself (r_0) will equal unity, and its coefficient of correlation (r_t) with the tth member following will necessarily equal its coefficient (r_{-t}) with the tth member preceding.

Any concrete instance of an experimentally obtained chance series we shall regard as a *model* of empirical processes which are structurally similar to it. As the basis of the present investigation we take three models of purely random series and call them the first, second, and third basic series. These series are based on the results obtained by the People's Commissariate of Finance in drawing the numbers of a government lottery loan. For the first basic series, we used the last digits of the numbers drawn; for the second basic series, we substituted 0 for each even digit and 1 for each odd digit; the third basic series was obtained in the same way as the second, but from another set of numbers drawn.[8]

Let us pass to the coherent series. Their origin may be extremely varied, but it seems probable that an especially prominent role is played in nature by the process of *moving summation* with weights of one kind or another; by this process coherent series are obtained from other coherent series or from incoherent series. For example, let causes $\cdots x_{i-2}, x_{i-1}, x_i, \cdots$ produce the consequences $\cdots y_{i-2}, y_{i-1}, y_i, \cdots$, where the magnitude of each consequence is determined by the influence, not of one, but of a number of the preceding causes, as for instance, the size of a crop is determined, not by one day's rainfall, but by many. If the influence of causes in retrospective order is expressed by the weights $A_0, A_1, A_2, \cdots A_{n-1}$, then we shall have

$$(1) \quad \begin{cases} y_i = A_0 x_i + A_1 x_{i-1} + \cdots + A_{n-1} x_{i-(n-1)}, \\ y_{i-1} = \qquad\quad A_0 x_{i-1} + \cdots + A_{n-2} x_{i-(n-1)} + A_{n-1} x_{i-n}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

[7] I venture to propose this name because it seems to me that it truly expresses what is intended, namely, the existence of some connection between the elements or parts of a thing (for example, of a series), but not a connection between this thing as a whole and another.

[8] The tables giving these series and seven others derived from them will be found in the original paper (*loc. cit.*, pp. 57–64) and are not repeated here.

Each of two adjacent consequences has one particular cause of its own, and $(n-1)$ causes in common with the other consequence. Because the consequences possess causes in common there appears between them a correlation even though the series of causes are incoherent. When all the weights are equal (*simple* moving summation) the coefficient of correlation expresses the share of the common causes in the total number of independent causes on which the consequences

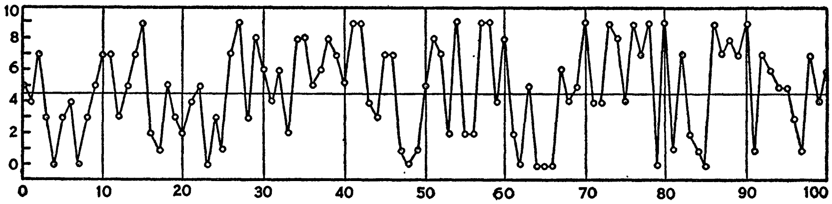

FIGURE 1.—The first 100 terms of the first basic series.

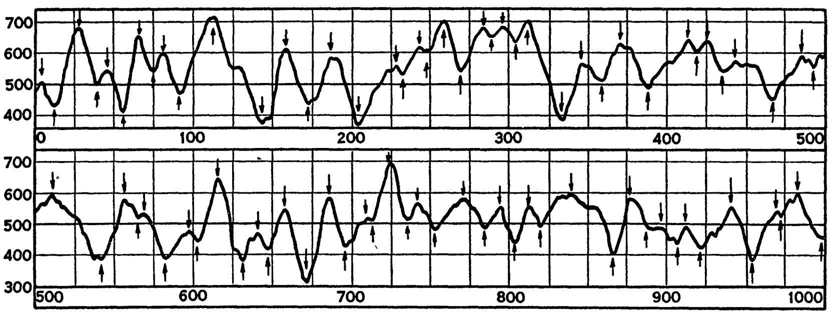

FIGURE 2.—The first 1000 terms of Model II.

depend (as has long been known from the theory of the experiment of Darbishire); then

$$r_0 = 1, r_1 = r_{-1} = \frac{n-1}{n}, r_2 = r_{-2} = \frac{n-2}{n}, \cdots, r_{n-1} = r_{-(n-1)} = \frac{1}{n},$$

further coefficients being equal to zero. By taking a ten-item moving summation of the first basic series, Model I was obtained.[9] A small section of Model I is plotted in Figure 3 with an index[10] of the English

[9] In addition, 5 was added to each sum. This does not change the properties of the series. Neither does it make any difference as to the method of numbering consequences in comparison with the scheme used in formula (1). At the outset of the work, it seemed to be technically more convenient to give the consequence the same number as the earliest cause and not the latest. Thus, for example, for Model I,

$$y_0 = x_0 + x_1 + x_2 + \cdots + x^9 + 5.$$

[10] Dr. Dorothy S. Thomas, "Quarterly Index of British Cycles," in W. L. Thorp, *Business Annals*, New York, 1926, p. 28.

business cycles for 1855–1877 in juxtaposition—an initial graphic demonstration of the possible effects of the summation of unconnected causes.

In turn the consequences become causes. Taking a ten-item moving summation of Model I, we obtained the 1000 numbers of Model II. Performing a two-item moving summation twelve times in succession on the third basic series,[11] we obtained the 1000 numbers of Model IVa. First and second differences of Model IVa give Models IVb and IVc respectively (See Figure 4). Furthermore, the application of scheme (1) to the second basic series gives[12] Model III if the weights used are

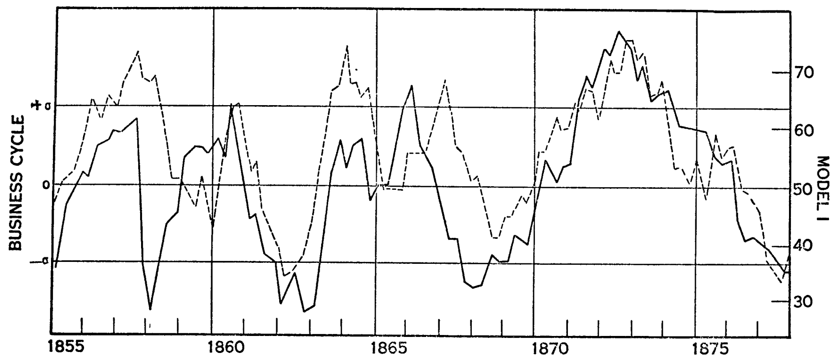

FIGURE 3.————An index of English business cycles from 1855 to 1877; scale on the left side. ---------Terms 20 to 145 of Model I; scale on the right side.

10^4 times the ordinates of the Gaussian curve taken at intervals of 0.1σ. Because this model was very smooth it appeared sufficient to use only the 180 even members out of the 360 items (see Figure 11 under the numbers 0, 2, 4, $\cdots$ 358). Model IIIa—the last one—is 10^4

[11] It actually was computed by applying the scheme (1) to the third basic series with the weights 1, 12, 66, 220, 495, 792, 924, 792, $\cdots$, 12, 1, because s-fold simple summation of two items is equivalent, as can be shown easily, to direct summation with the weights $C_0^s, C_1^s, C_2^s, \cdots, C_s^s$, (where C_k^s is the number of combinations of s things taken k at a time.

[12] The exact values of Model III could be obtained by multiplying the corresponding items of the basic series by the exact values of the function $10^4 \exp\{-\frac{1}{2}(0.1t)^2\}/\sqrt{2\pi}$, for integral values of t. This function was the basis of obtaining the 4th differences of Model III. Approximate values of Model III were found by using a set of weights composed of 95 numbers corresponding to the values of the above function for integral values of t from -47 to $+47$, with the numbers less than 1 rounded off to the nearest tenth and numbers greater than 1 to whole units. The numbers of the basic series were written on a ribbon which we slid along the column of weights. Inasmuch as the basic series consisted of zeros and ones, all of the computations were plain additions. For Model IVa, a ribbon with holes in the place of unities was constructed.

times the 4th differences of the numbers of Model III from the 7th to the 97th.[13]

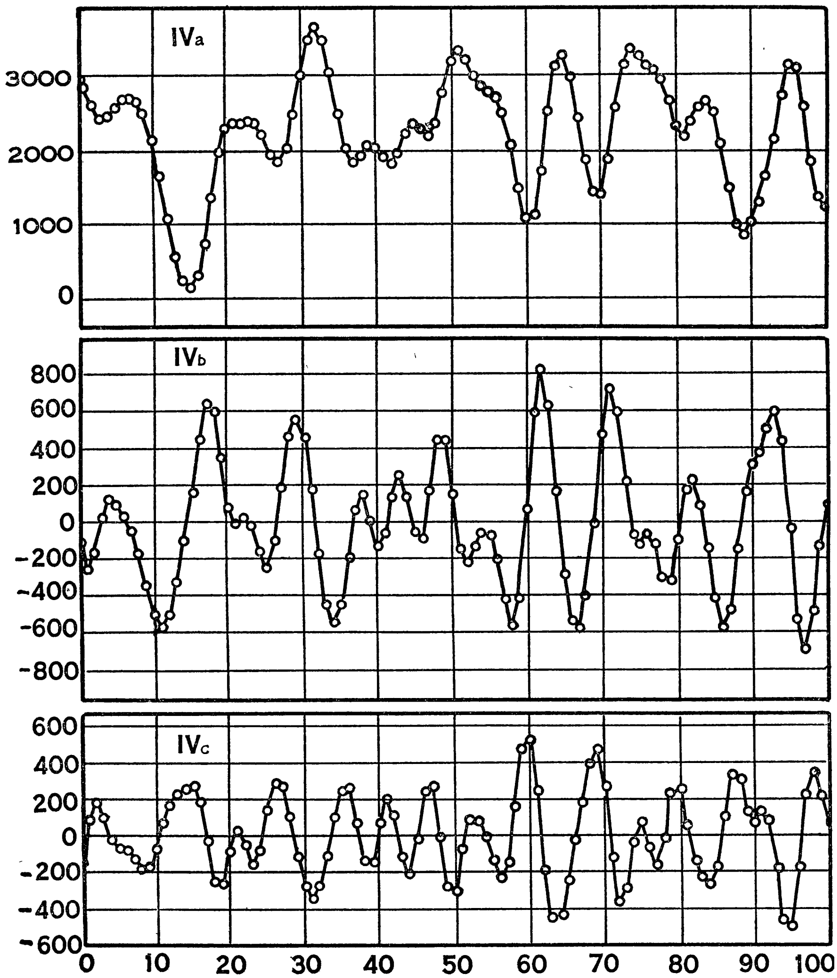

FIGURE 4.—The first 100 terms of Models IVa, IVb, and IVc.

We could not be satisfied by a smaller number of models because it was necessary to observe their various properties and to have illustra-

[13] For the calculation of these differences the accuracy with which we determined the items of Model III was not sufficient, so the following method was used: It is easy to see that the nth order differences of the items of the series obtained by scheme (1) are equivalent to those computed by the same scheme but applying weights equal to the differences of the original weights (keeping in mind that the series of original weights is extended at both ends with zeros).

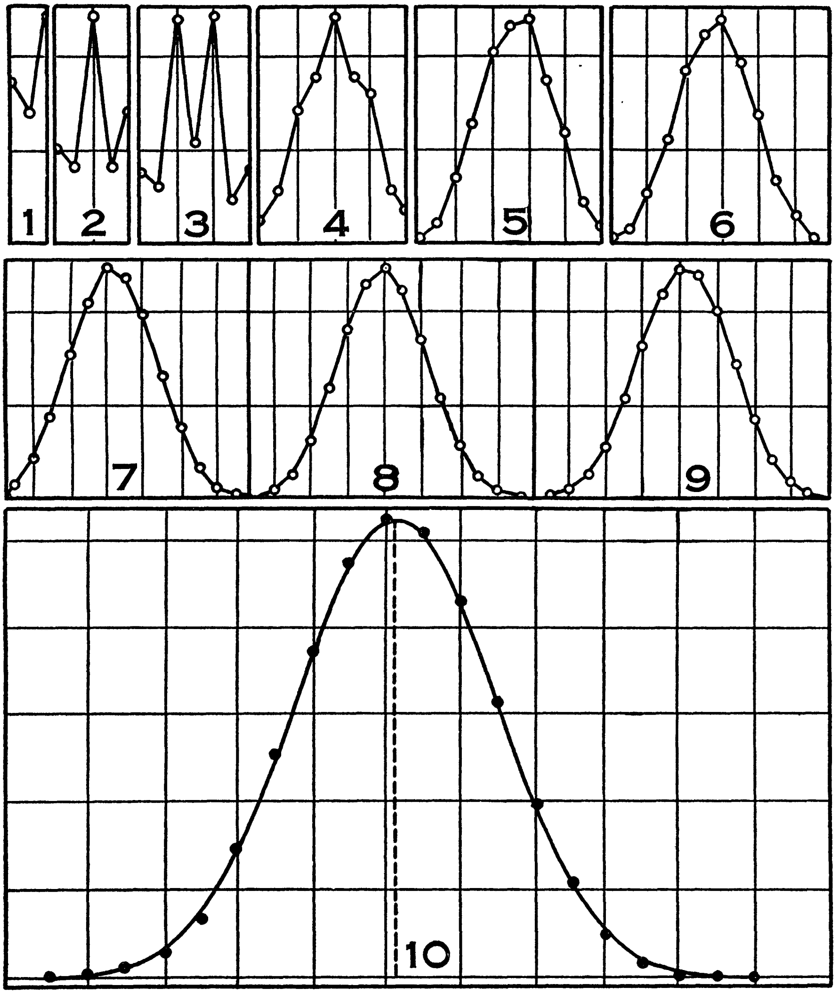

FIGURE 5.—An example of the crossing of random weights. The weights of the causes for each of 10 successive summations are shown, i.e., $A_k'^{(1)}$, $A_k'^{(2)}$, $\cdots$, $A_k'^{(10)}$. See Appendix, Section 1.

With the help of S. Pineto's *Tables de logarithmes vulgaires a dix decimales*, St. Petersburg, 1871, the values of the function

$$\exp\left\{-\tfrac{1}{2}(0.1t)^2\right\}/\sqrt{2\pi}$$

were obtained to ten decimal places for integral values of t from 0 to 44; this series was completed by using Sheppard's tables, and the differences of the entire series up to and including the 4th differences were taken. Multiplying the latter by 10^8 and expressing the result in integers, we obtained weights with the help of which—and by using scheme (1)—the values of $10^4\ \Delta^4 y_{III}$ were obtained from the second basic series.

tions for the elucidation of the different aspects of the problem. We could not aspire to imitate nature in forming a set of weights; still, in the course of the work, we have come across an exceptionally curious circumstance. First, each multifold simple summation of n items at a time gives a set of weights which approaches the Gaussian curve as a limit. In the Appendix, Section 1, there is given the instance of a tenfold summation of three items at a time with the weights chosen absolutely at random for each successive summation. The ten con-

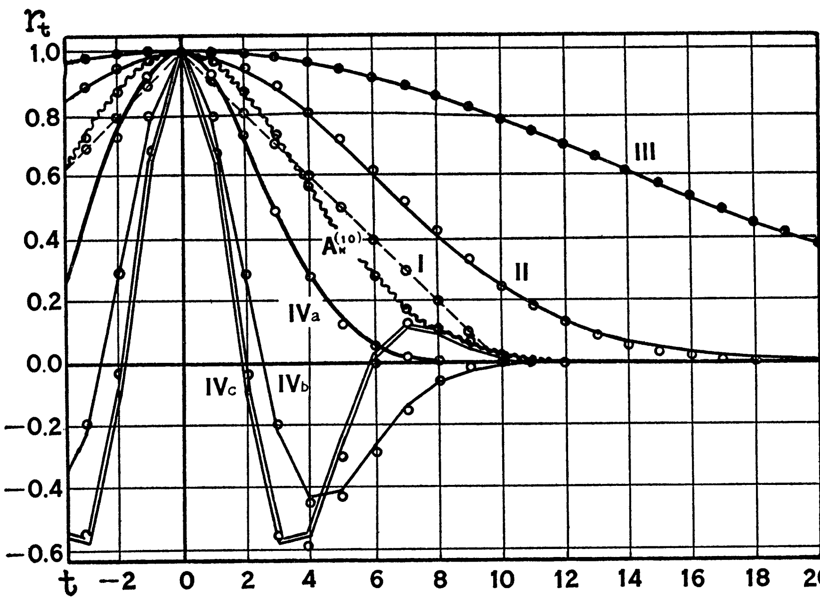

FIGURE 6.—oooo The correlational functions of Models I–IVc, and of the scheme of the crossing of the chance weights ($A_k'^{(10)}$).

} Corresponding Gaussian curves and the reduced differences of the ordinates of the Gaussian curve.

secutive sets of weights are depicted in Figure 5. It is easily seen how they gradually become more and more like the Gaussian curve, and for the tenth summation the weights approach the Gaussian curve very closely.

This is far from being a chance result. From further considerations (Appendix, Section 1) we find that we have here actually encountered a law which, under certain conditions, must necessarily realize itself in the chaos of random entanglements and crossings of endless numbers of series of causes and consequences. The problem is specially important for the reason that the correlational function of a derived series is defined entirely by the respective weights-function. It is possible to prove (see Appendix, Sections 1, 2, and 3) that if the series of weights follows the Gaussian curve, the correlational function of the resulting consequence series is capable of being expressed by a similar curve with a greater or smaller degree of approximation. For the series of consequences proportionate to the increments of the cause—that is, the differences of order k of the series of causes—the correlational function can be represented by the series of the differences (of order $2k$) of the ordinates of the Gaussian curve. It could not be by chance that the correlational function of all of our models, with the exception of the most elementary one (Model I), belong to one of the two types mentioned (see Figure 6). No exception is found in the correlational

TABLE 1

Distance between terms	Correlation coefficients with random weights	Ordinates of Gaussian curve	Differences
t	r_t	R_t	$r_t - R_t$
0	1.000	1.000	0.000
1	0.965	0.965	0.000
2	0.868	0.866	+0.002
3	0.727	0.723	+0.004
4	0.567	0.562	+0.005
5	0.410	0.407	+0.003
6	0.275	0.274	+0.001
7	0.171	0.171	0.000
8	0.097	0.100	−0.003
9	0.051	0.054	−0.003
10	0.024	0.027	−0.003
11	0.011	0.013	−0.002
12	0.004	0.006	−0.002
13	0.001	0.002	−0.001
14	0.000	0.001	−0.001

function for the series of consequences of the 10th order obtained in the course of the crossing of the random weights in the example mentioned above. The values of these correlation coefficients (r_t), together with the ordinates of the corresponding Gaussian curve (R_t), are given in Table 1 (for the calculation see Appendix, Section 2).

III. THE UNDULATORY CHARACTER OF CHANCE SERIES; GRADUALITY AND FLUENCY AS TENDENCIES

Our models, representing several sets of experiments, give an inductive proof of our first thesis, namely, *that the summation of random causes may be the source of cyclic, or undulatory processes.*[14] It is, however

[14] The definition of the business cycle as being a process (not necessarily periodic) characterized by successive rises and falls, is given by *W. C. Mitchell* in Introduction to *W. L. Thorp, Business Annals,* New York, 1926, pp. 32–33.

not difficult to determine the reason why it must be so inevitably. We shall first observe a series of independent values of a random variable. If, for the sake of simplicity, we assume that the distribution of probabilities does not change, then, for the entire series, there will exist a certain horizontal level such that the probabilities of obtaining a value either above or below it would be equal. The probability that a value, which has just passed from the positive deviation region to the negative, will remain below at the subsequent trial is $\frac{1}{2}$; the probability that it will remain below two times in succession is $\frac{1}{4}$; three times $\frac{1}{8}$; and so on. Thus the probability that the values will remain for a long time above the level or below the level is quite negligible. It is, therefore, practically certain that, for a somewhat long series, the values will pass many times from the positive deviations to the negative and vice versa. Let us designate as a *half-wave* a portion of the series in which the deviation does not change sign. Thus, for 1000 numbers of the third basic series we find 540 half-waves (instead of the theoretically expected 500). Taking from this number the first 512 half-waves we find among them a number of half-waves of the length 1,2, etc. In Table 2 the actual (n_i') and theoretical[15] (n_i) frequencies for half-waves of various lengths are shown. That the observed series is consistent with the theoretical series can be found by the calculation of the χ^2 criterion of goodness of fit.[16]

TABLE 2

Length of half-wave	Actual frequency	Theoretical frequency
i	n'_i	n_i
1	261	256
2	137	128
3	65	64
4	29	32
5	14	16
6	4	8
7	1	4
8 and more	1	4
Total	512	512

If a variable can have more than two values and if, in a certain interval of a more or less considerable length, it happens to remain above

[15] L. von Bortkiewicz, *Die Iterationen*, 1917, Formel 75, p. 99.

[16] We find, indeed,

$$\chi^2 = \sum \frac{(n_i' - n_i)^2}{n_i} = 7.78,$$

the corresponding probability being $P = 0.35$; see *Tables for Statisticians and Biometricians*, ed. by K. Pearson, Part I, Table XII.

(or below) its general level, then in that interval it will have a temporary level about which it almost certainly will oscillate. Thus on the waves of one order there appear superimposed waves of another order.

The unconnected random waves are usually called irregular zigzags. A correlation between the items of a series deprives the waves of this characteristic and introduces into their rising and falling movements an element of *graduality*. In order to make the reasoning more concrete, let us consider a series obtained from an incoherent series by means of a ten-item moving summation. Our Model I will be used as the example. Any items of this model separated from each other by more than 9 intervals (as, for example, the values $y_0, y_{10}, y_{20}, \cdots$) are not correlated with each other and consequently form waves of the above considered type, i.e., irregular zigzags. But if we consider the entire series, we shall certainly find gradual transitions from the maximum point of a wave to its minimum and vice versa, since the correlation between neighboring items of the series makes small differences between them more probable than large ones. This we find to be true for all of our models.

We must distinguish between the *graduality* of the transitions and their *fluency*. We could speak about the absence of the latter property if a state of things existed where there would be an equal probability for either a rise or a fall after a rise as well as after a fall. If fluency were missing we should obtain waves covered by zigzags such as we find in Model I (see Figure 3).

For example, we have for Model I,

$$\begin{aligned} y_0 &= 5 + x_0 + x_1 + x_2 + \cdots + x_9, \\ y_1 &= 5 \qquad + x_1 + x_2 + \cdots + x_9 + x_{10}, \\ y_2 &= 5 \qquad\qquad + x_2 + \cdots + x_9 + x_{10} + x_{11}, \\ &\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

from which we obtain

$$\begin{aligned} \Delta y_0 &= y_1 - y_0 = x_{10} - x_0, \\ \Delta y_1 &= y_2 - y_1 = x_{11} - x_1, \\ &\cdots\cdots\cdots\cdots . \end{aligned}$$

Thus we see that the adjacent first differences do not have any causes in common, and hence are not correlated. The same applies to differences which are further apart, with the exception of such as $y_1 - y_0 = x_{10} - x_0$ and $y_{11} - y_{10} = x_{20} - x_{10}$. The series of differences is almost incoherent and hence the waves will be covered by chaotically irregular zigzags such as we find in Model I.

Let us assume further that adjacent differences are positively correlated. Then, in all probability, after a rise another rise will occur, after a fall a further fall; a steep rise will have the *tendency* to continue with the same steepness, a moderate one with the same moderateness. So small sections of a wave will tend to be straight lines; and the greater the coefficient of correlation between adjacent differences the closer the sections approximate straight lines.[17]

Correlation between second differences plays an analogous role. The greater this correlation coefficient, the greater the tendency toward the preservation of the constancy of the second differences. Over more or less considerably long intervals a series with approximately constant second differences will tend to approximate a second-degree parabola as all "good" curves do. In Table 3 are given, for Models, I, II, and III, the values of the correlation coefficients between the adjacent items of the series (r_1), between the adjacent first differences ($r_1^{(1,1)}$), and between the adjacent second differences ($r_1^{(2,2)}$). The coefficients were calculated by the formulas of the Appendix, Section 1. As we go from the first basic series to Model I and then to Models II and III, we find progressive changes in their graphic appearance (see Figures, 3, 2, 8, and 11 respectively). These changes are produced at first by the introduction and then by the growth of graduality and of fluency in the movements of the respective chance waves. The growth of the degree of correlation between items (or between their differences) as we go from the first basic series to Model I, etc. (see Table 3) corresponds to the changes in the graphic appearance of our series.

TABLE 3

Model	Coefficient of correlation between:		
	Terms	First differences	Second differences
	r_1	$r_1^{(1,1)}$	$r_1^{(2,2)}$
I	0.9	0.0	−0.5
II	0.985	0.85	0.0
III	0.9975	0.9925	0.9876

IV. EMPIRICAL EVIDENCE OF THE APPROXIMATE REGULARITY OF CHANCE WAVES

Our first thesis, that is, the demonstration of the possibility of the appearance of undulatory processes of a more or less fluent character as the result of the summation of random causes, may be considered

[17] The term *tendency* is used here in a strict sense. To each equation of regression (giving the value of the conditional mathematical expectation of a variable as a function of some other variable) there corresponds an approximate equation between the variables themselves. The closer to unity the absolute value of the coefficient of correlation lies, the greater is the probability that this functional relationship will be maintained within the limits of the desired accuracy; i.e., the stronger will be the *tendency*.

as practically proved. However, our second thesis, that is, the demonstration of *the approximate regularity of the waves*, offers considerably greater difficulties. Again we shall begin with the inductive method.

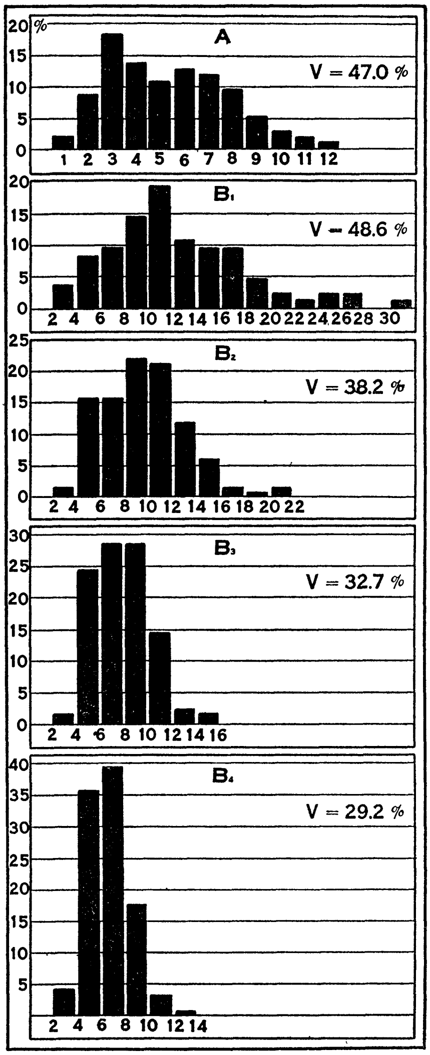

FIGURE 7.—The frequency distributions of the lengths of waves and half-waves: *A*, Business cycles of 12 countries, not including England (Mitchell); B_1 to B_4, Models II, IVa, IVb, and IVc respectively.

In Figure 2 are plotted 1000 points of Model II; a continuous line has been passed through them, which, because of the small scale, seems to be a comparatively fluent curve. One can distinguish on the curve waves of different orders—even down to insignificant zigzags, of which a number are not apparent on the graph because of their minuteness. The maxima and minima having been listed, together with the length of their half-waves and amplitudes, we have found that, since an empirically descriptive point of view, in its very nature, permits only approximate solutions,[18] it was legitimate to draw a boundary between waves and *ripples:* maxima and minima with amplitudes of ten units or less being discarded as ripples. The remaining maxima and minima are indicated by arrows in Figure 2. The distribution of the lengths of the 83 half-waves for Model II is given graphically in Figure 7 (B_1). Figure 7 includes the distribution (A) of the lengths of 93 cycles of economic life for 12 countries outside of England, as given by Mitchell.[19] The coefficient of variation for the latter is 47.0%[20] as compared to 48.6% for Model II. Thus we find variation of the same degree in the two distributions. The distributions for Models IVa, IVb, and IVc are also shown in Figure 7. The average lengths of waves are 9.23, 7.36 and 6.15, while the coefficients of variation are 38.2%, 32.7% and 29.2% respectively. In general appearance these last three distributions are similar to the first two, although the last three have less variation, in spite of the fact that for Models IV, a, b, and c, the data are taken without discarding the ripples. Our models being based on some a priori schemes, it appears quite likely that some day it will be possible to calculate the mathematical expectation and variability of the distances between maxima and minima. In this respect, therefore, the chance waves in coherent series must be subject to some kind of regularity, the regularity of this type being observed even in the chaotic zigzags of purely random series.[21]

We are interested, however, in a different aspect of the problem. The attempt of Mitchell to deny the periodicity of business cycles is a result of his tendency to stick to a purely descriptive point of view. The means of description which he uses and which we tried to imitate for our models are far too crude. If we try to apply the same method to a sum of two or three sinusoids the result would be approximately the same. Those investigators of economic life are right who believe in their acumen and instinct and subscribe to at least an approximate correctness in the concept of the periodicity of business cycles. Let us

[18] Cf. E. Husserl, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Halle a.d.S., 1922, § 74: *Deskriptive und exakte Wissenschaften*, p. 138–139.

[19] W. L. Thorp, *Business Annals*, Introduction by W. C. Mitchell, p. 58.

[20] *Ibid.*

[21] Cf. L. von. Bortkiewicz, *Die Iterationen*, 1917.

again examine Model II (Figure 2). In many places there are, apparently, large waves with massive outlines as well as smaller waves lying, as it were, over them; sometimes these are detached from them, sometimes they are almost completely merged into them. For example, at the beginning of Figure 2, three waves of nearly equal length are apparent, that is, from the first to the third minimum, from the third to the fifth, and from the fifth to the sixth. Upon these waves smaller

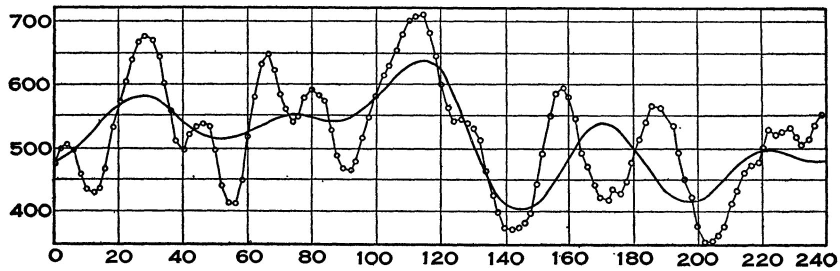

FIGURE 8.—o-o-o The first 120 even terms of Model II. ——— Sum of the first five harmonics of Fourier series: $y=518.14-20.98 \cos(2\pi t/240)+50.02 \sin(2\pi t/240)+17.30 \cos(2\pi t/120)-3.16 \sin(2\pi t/120)-10.93 \cos(2\pi t/80)+35.66 \sin(2\pi t/80)+17.18 \cos(2\pi t/60)-21.92 \sin(2\pi t/60)-38.53 \cos(2\pi t/48)-3.65 \sin(2\pi t/48)$.

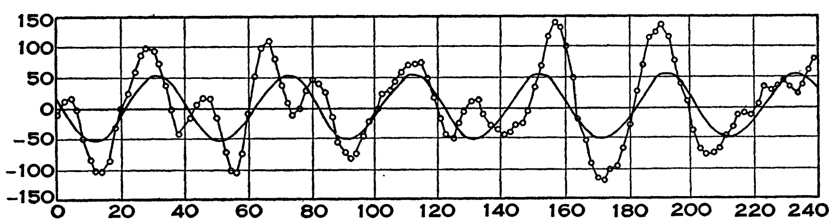

FIGURE 9.—o-o-o The deviations of Model II from the sum of the first five harmonics of Fourier series. ——— 6th sinusoid: $y=12.98 \cos(2\pi t/40)-51.50 \sin(2\pi t/40)$.

ones can be seen having also approximately equal dimensions. A careful examination of the graphs of our models will disclose to the reader a number of places where the approximate equality of the length of the waves is readily apparent. If we had a much shorter series, such as a series offered by the ordinary statistics of economic life with its small number of waves, we should be tempted to consider the sequence as strictly periodic, that is, as composed of a few regular harmonic fluctuations complicated by some insignificant casual fluctuations. For instance, let us consider two sections of Model II, lying one directly above the other in Figure 2, namely, the section from item 100 to item 250 and the one from 600 to 750. The similarity between the waves in these sections is apparent.

The accuracy of the above deduction is limited by the imperfection of a visual impression. To eliminate this shortcoming, let us analyze one or two sections of our models harmonically by means of Fourier's analysis. This has been done for a section of 240 points of Model II and the 360 points of Model III. Because of the great fluency of these series it was sufficient to use only the even-numbered ordinates (i.e., 0, 2, · · · 238, and 0, 2, · · · 358, respectively), thus saving some computation. The results for the 120 points of Model II are shown in Figures 8, 9 and 10, those for the 180 points of Model III in Figures 11 and 12.

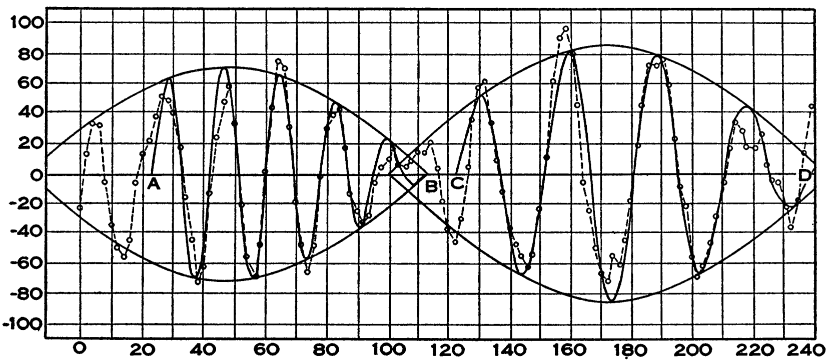

FIGURE 10.—o-o-o The deviations of Model II from the sum of six harmonics.

$$A\text{———}B\colon y_I = 71 \sin \frac{2\pi}{264}(t + 18) \sin \frac{2\pi}{18}(t - 24).$$

$$C\text{———}D\colon y_{II}\ 85 \sin \frac{2\pi}{288}(t - 100) \sin \frac{2\pi}{14\frac{2}{3}}(t - 122\tfrac{1}{3})$$

First let us consider Model II. In Figure 8 the sum of the first five sinusoids of the Fourier series are shown, while in Figure 9 the deviations from that sum are shown together with the sixth sinusoid. It is known, of course, that practically any given curve can be represented by a sum of a series of sinusoids provided a large enough number of terms is taken. It is not for every empirical series, however, that we can obtain such a significant correspondence and such a sharply expressed periodicity with a comparatively small number of harmonics. The approximately regular waves which were apparent even in the crude series are much more distinct now when they are isolated by deducting the sum of the first five harmonics. Of course, we cannot assert that the rest of Model II would follow the same periodicity, but, for our purposes, it is sufficient that successive waves should maintain an approximate equality of length for six periods. This hardly can be considered to be a chance occurrence; the explanation of such an effect must be found in the mechanism of the connection of the random values.

The deviations from the sum of six harmonics are plotted in Figure 10 together with the corresponding fluent curves. These curves are obtained as interference waves of two sinusoids with equal amplitudes and approximately equal periods. In other words, such a curve can be represented as the product of two sinusoids or as a sinusoid with an amplitude also changing along a sinusoid. These *bending sinusoids*

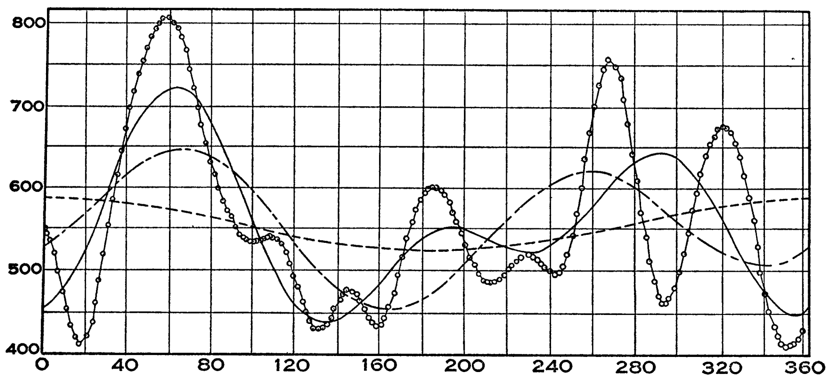

FIGURE 11.—o-o-o The first 180 even terms of Model III. ----$y_I=554.8+31.79\cos(2\pi t/360)+3.40\sin(2\pi t/360)$. ------ $y_{II}=y_I-58.82\cos(2\pi t/180)+46.63\sin(2\pi t/180)$. ——— $y_{III}=y_{II}-75.36\cos(2\pi t/120)+0.61\sin(2\pi t/120)$.

separate on the graph the regions which place our empirical series in a definite *regime*.[22] Over a large part of the first region the regime is maintained for three or four periods with a correspondence that is much greater than could reasonably be expected between an analytical curve and a random series. At the beginning and end of a region the regime is broken. The point where a bending sinusoid cuts the axis of abscissas is the *critical point*. After this point a regime is replaced by another regime of the same type, but having different parameters. Throughout the greater part of the second region, as in the first, the regime is quite well sustained.[23]

[22] The term *regime* has been borrowed for the purposes of theoretical statistics from hydrography by N. S. Tchetverikov. See his work: "Relation of the Price of Wheat to the Size of the Crop," *The Problems of Economic Conditions*, Vol. 1, Issue 1, Moscow, 1925, p. 83.

[23] The parameters of a regime,

$$y = A\sin[(360°/L)(x-a)]\sin[(360°/l)(x-b)]$$

are easy to determine by means of graphical construction after a few trials. It is also possible to make corrections, using the method of least squares, but in our case we did not think it necessary.

If a result like the foregoing is not due to chance, a much better proof could be expected from an analysis of Model III for which the correlation between the elements is greater than for Model II. In Figure 11 the even-numbered points from 0 to 358 of Model III are plotted together with the first harmonic of the Fourier series, the sum of the first two, and the sum of the first three sinusoids. Instead of the six sinusoids needed for Model II, only three are here necessary for our purposes. The deviations from these are shown in Figure 12. Three regions are apparent with a change of regimes at the critical

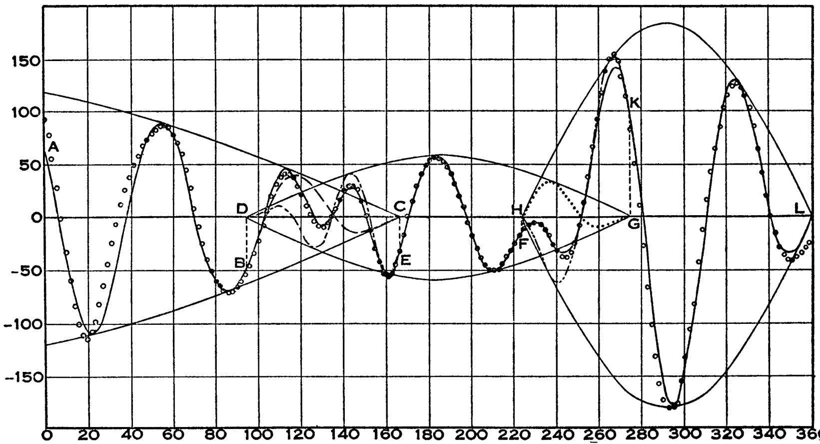

FIGURE 12.—oooo Deviations of Model III from the sum of the first three harmonics. *A*——*B*----*C*, Regime I: $y_I = 136 \sin [2\pi(t-167)/960] \sin [2\pi(t-39)/64]$. D-·-·-*E*, Regime II′: $y_{II}' = 58 \sin [2\pi(t-94)/360] \sin [2\pi(t-98)/36]$. *E*——*F*····*G*, Regime II″: $y_{II}'' = 58 \sin [2\pi(t-94)/360] \sin [2\pi(t-170)/54.4]$. *H*-··-··-*K*——*L*, Regime III: $y_{III} = 182 \sin [2\pi(t-222)/276] \sin [2\pi(t-250.6)/59.6]$. *B*——*E*; $y = y_I + y_{II}'$. *F*——*K*: $y = y_{II}'' + y_{III}$.

points. In addition we find one more regularity: to the overlapping parts of the said regions corresponds every time the partial superposition of the regimes, i.e., the algebraical addition of the respective curves.

Let us try now to summarize our observations in the following tentative and hypothetical manner:

The summation of random causes generates a cyclical series which tends to imitate for a number of cycles a harmonic series of a relatively small number of sine curves. After a more or less considerable number of periods every regime becomes disarranged, the transition to another regime occurring sometimes rather gradually, sometimes more or less abruptly, around certain critical points.

V. THE TENDENCY TO SINUSOIDAL FORM

In addition to the tendencies towards graduality and fluency (that is towards linear and parabolic forms for small sections) we find a third tendency, namely, the tendency toward a sinusoidal form.

Let $y_i, y_{i+1}, y_{i+2}, \cdots$ be the ordinates of a sinusoid. Then it is always true that

$$\Delta^2 y_i = -a y_{i+1}, \tag{2}$$

where $\Delta^2 y_i = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i)$, that is, the ith second difference of the series.

Conversely, it can easily be proved that the function defined by an equation of the form (2) in case $0 < a < 4$ must be a sinusoid.[24] Now, if there is a high correlation between the second differences ($\Delta^2 y_i$) and the ordinates (y_{i+1}) of a series, then equation (2) will be approximately true and there will exist a tendency toward a sinusoidal form in the series. The closer the correlation coefficient between $\Delta^2 y_i$ and y_{i+1}, denoted by us by $r_1^{(2,0)}$, is to -1, the more pronounced (or strong) is the tendency to a sinusoidal form.

A tendency toward either linear or parabolic forms cannot appear in a very large section of a coherent series because it would disrupt its cyclic character. The accumulation of deviations necessarily destroys every linear or parabolic regime even though the respective correlations are very high. After a regime is disrupted the new section will have a new, let us say a parabolic, regime (i.e., a regime of parabolas with different parameters). This process continues throughout the entire series, so that each coherent series of the type considered here is patched together out of a number of parabolas with variable parameters whose variations generally cannot be foreseen.

A sinusoidal regime is also bound to disrupt gradually, this being a property which distinguishes every tendency from an exact law. But under favorable conditions the sinusoidal tendency can be maintained over a number of waves without contradicting the basic property of a coherent series. In order to obtain a result of this kind it is necessary that the respective correlations be sufficiently high. But, as a matter of fact, $r_1^{(2,0)}$ for Model II is approximately the same as for Model I (-0.315 and -0.316), while Model III with its great smoothness has an $r_1^{(2,0)}$ less than that of Model IVa (-0.578 as compared to -0.599). It seems, however, to be very probable that this criterion is insufficient just because we have to deal here not with one sinusoid but with a whole series of sinusoids having different periods. Equation (2), of

[24] The condition $0 < a < 4$ is always satisfied in our case since $a = 2(1 - r_1)$ where r_1 is a correlation coefficient between the adjacent terms of the series (between y_i and y_{i+1}). See Appendix, Section 4.

course, is true only for a single sinusoid and cannot be applied to a sum of sinusoids.

To find an instance more apt to illustrate the tendency in question, let us consider the differences of various orders for Model III, the series best adapted for such purposes. If a curve is represented by a sum of sinusoids, then the differences of all orders are sums of sinusoids having waves of the same periods as the curve. The higher the order of the difference, the more pronounced are the shorter periods, since the differencing process weights the shorter periods as against the longer ones. Thus, by applying the formulas of the Appendix, Section 1, we find

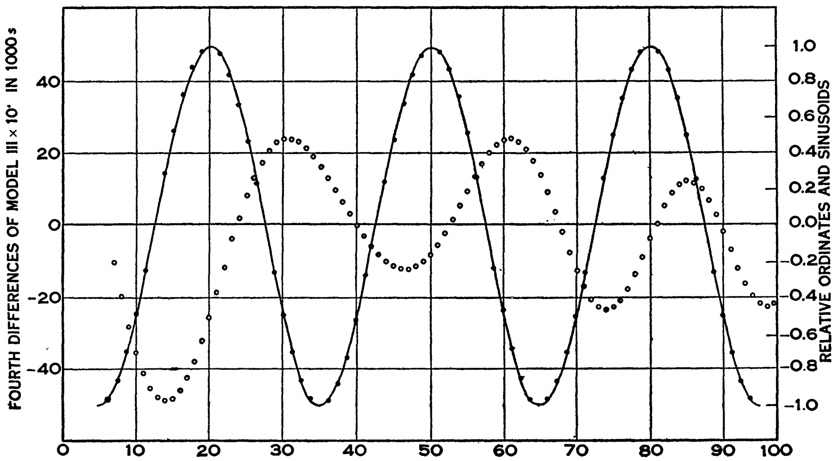

FIGURE 13.—oooo The fourth differences of Model III$\times 10^4$. —·—·– Their relative ordinates, together with the sinusoid.

that the coefficients of correlation, $r_1^{(2,0)}$, for Model III itself, and for its first, second, third, and fourth differences, are -0.5781, -0.7756, -0.8462, -0.8830, and -0.9057, respectively. The following considerations will show us to what extent the simple sinusoidal regime is maintained at least over small portions of the last series.

Let us determine the highest (or lowest) point of a typical wave (A or B, respectively, in Figure 14) as the apex of a second-degree parabola which passes through the three highest (or the three lowest) points of the wave. Then, let us draw a horizontal line bisecting the distance between the highest and lowest points of the wave. Further, let us denote the point where this horizontal line crosses the straight line joining the two points between which the horizontal line passes as C. This point divides AB into two quarter-waves, AC and BC. For each of these, let us make the following construction: Dividing the base line DC (D having the same abscissa as A) into six equal parts, we obtain seven points corresponding to 0°, 15°, 30°, 45°, 60°, 75°, and 90°. At the five central points construct perpendiculars and extend them to the parabola fitted to the three empirical points (interpolated according to Newton's formula). These perpendiculars are the ordinates of an empirical half-wave and, if we divide through by the maximum ordinate AD, we obtain the relative ordinates y_{15}, y_{30}, y_{45}, y_{60}, and y_{75}. If our wave is a sinusoid, these relative ordinates will equal the sines of 15°, 30°, 45°, 60°, and 75°, respectively. The empirical relative ordinates for the 12 quarter-waves of $\Delta^4 y_{III}$ are shown by black dots around the regular sinusoid of Figure 13, while the relative ordinates of the first, second, etc., quarters of every empirical wave are shown on the first,

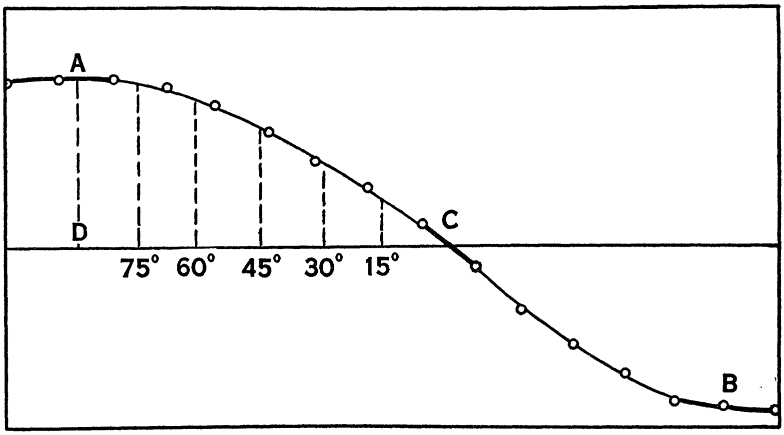

FIGURE 14.—A Scheme for Calculation of the Relative Ordinates.

second, etc., quarters of the sinusoid. The points can hardly be distinguished from the curve. Thus the tendency to a sinusoidal form is shown rather distinctly. If we compute the arithmetic averages of the relative ordinates having the same abscissa (e.g., $\bar{y}_{15}=1/12(y_{15}^{(1)}+y_{15}^{(2)} + \cdots +y_{15}^{(12)})$ and compare them with the corresponding sines (e.g., sin 15°), we shall see that the deviations are less than $\frac{1}{2}$ in the second decimal place (see Table 4). The agreement is, therefore, close enough

TABLE 4

Phase-angle (α)	15°	30°	45°	60°	75°
$\bar{y}_\alpha$	0.258	0.496	0.703	0.863	0.964
Sin α	0.259	0.500	0.707	0.866	0.966
Deviations	−0.001	−0.004	−0.004	−0.003	−0.002

to be considered as the clear manifestation of the tendency toward a sinusoidal form, and thus displays once more the ability of chance waves to simulate regular harmonic oscillations.

VI. ON THE PSEUDO-PERIODIC CHARACTER OF THE EMPIRICAL CORRELATIONAL FUNCTION[25]

As a further illustration of the sinusoidal tendency, I shall consider here a chance series satisfying, to a rather high degree of approximation, the equation

$$\Delta^4 z_i - p\Delta^2 z_{i+1} - q z_{i+2} = 0, \tag{3}$$

corresponding, if treated as a precise one, to the sum of two sinusoids.

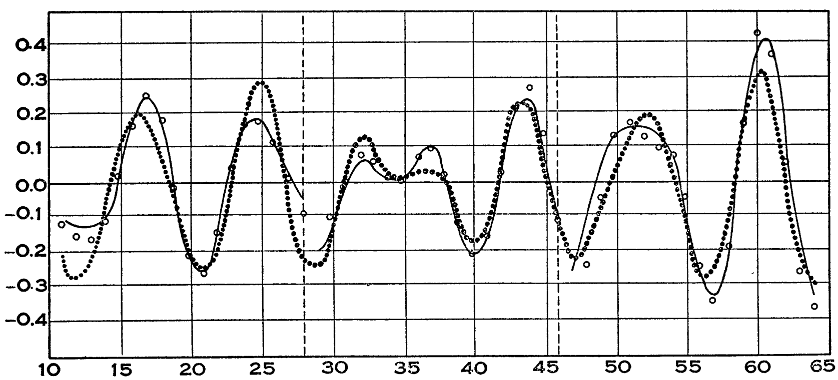

FIGURE 15.— o o o o The reduced empirical correlation function of Model IVc. ——— Sum of two sinusoids, separately for each of three intervals. ..... Sum of three sinusoids.

Let us denote the series in question by the symbols

$$\rho'_{11},\ \rho'_{12},\ \cdots,\ \rho'_{64},$$

the values of ρ'_t (see Figure 15) being given by the equation

$$\rho'_t = \sqrt{\frac{128 - t}{128}}\,\rho_t, \tag{4}$$

where ρ_t is the empirical correlation coefficient between the terms (y_i, y_{i+t}) of the series made up by 128 items of Model IVc. As the values of ρ_t have been calculated from very different numbers of items varying from 117 (that is, 128−11) to 64 (that is, 128−64), the reduction

[25] Eugen Slutzky, "On the Standard Error of the Correlation Coefficient in the Case of Homogenous Coherent Chance Series" (in Russian, with English summary). Transactions of the Conjuncture Institute, Vol. 2, 1929, pp. 94–98, 154.

by (4) has been thought useful in order to bring the respective standard deviations to approximate equality.

Before going further, the following remarks will be made. Let y_0, $y_1, \cdots$ be a stationary chance series. This implies that the mathematical expectation, $E(y_i)$, is a constant, that the standard deviation, σ, is a constant, and that the correlation coefficient, r_t, between y_i and y_{i+t}, is a function of t only. Then, putting, without loss of generality, $E(y_i)=0$, we shall have $\sigma^2=E(y_i^2)$ and $r_t=E(y_iy_{i+t})/\sigma^2$. This being the *theoretical* correlation coefficient, let us suppose that $r_t=0$ if $t>\omega$. Then the correlation coefficient, r'_u, between the empirical correlation coefficients, ρ_t and ρ_{t+u}, will be given by the equation

$$(5) \qquad r'_u = r(\rho_t, \rho_{t+u}) = \frac{\sum_{-\omega}^{\omega} r_t r_{t+u}}{\sum_{-\omega}^{\omega} r_t^2},$$

this formula being approximately correct if it be supposed (1) that ρ_t and ρ_{t+u} are calculated from the same number of values, n; (2) that n is sufficiently large; and (3) that $t>2\omega$, $n-t>\omega$, and $u>0$.[26]

Let us suppose now that the values of r'_u calculated from (5) may be held to be approximately true for the series of the reduced correlation coefficients $(\rho'_{11}, \rho'_{12}, \cdots \rho'_{64})$ defined above. Then we have to consider the following problem:

The series given (ρ'_t) being a chance series, there can exist no periodicity in the strict sense of the word. Its cyclical character being, however, obvious (see Figure 15), it may be asked whether the law of its composition from simple harmonics cannot be detected when its correlational function is known.

Let us try to solve this problem to the first approximation by supposing that our series can be duly approximated by the sum of two sinusoids with constant periods and varying amplitudes and phases. To this end, let us find the parameters of the regression equation which can be written in the form

$$(6) \qquad \Delta^4\rho'_u = p\Delta^2\rho'_{u+1} + q\rho'_{u+2} + \epsilon,$$

ϵ being the "error," and p and q being determined by the method of least squares. If we denote the correlation coefficients between the pairs of values

$$(\Delta^4\rho'_u, \Delta^2\rho'_{u+1}), (\Delta^4\rho'_u, \rho'_{u+2}), (\Delta^2\rho'_{u+1}, \rho'_{u+2}),$$

by r_{12}, r_{13}, r_{23} respectively, then, using the formulae (43)–(44) (Appendix, Section 1), we obtain

[26] Cf. Slutzky, *loc. cit.* in note 25, pp. 91–94.

$$(7) \qquad r_{12} = \frac{\Delta^6 r'_{-3}}{\sqrt{\Delta^8 r'_{-4} \Delta^4 r'_{-2}}}; \quad r_{13} = \frac{\Delta^4 r'_{-2}}{\sqrt{\Delta^8 r'_{-4}}}; \quad r_{23} = \frac{\Delta^2 r'_{-1}}{\sqrt{\Delta^4 r'_{-2}}},$$

where r'_{-u} $(=r'_u)$ is the correlation coefficient defined by the equation (5). These values are

$$\begin{aligned} r'_{-4} &= r'_4 = -\,0.761{,}874{,}30, \\ r'_{-3} &= r'_3 = -\,0.618{,}465{,}96, \\ r'_{-2} &= r'_2 = -\,0.013{,}793{,}10, \\ r'_{-1} &= r'_1 = -\,0.689{,}655{,}17, \end{aligned}$$

whence, using (7),

$$r_{12} = 0.945{,}847, \quad r_{13} = 0.760{,}844, \quad r_{23} = -\,0.919{,}999.$$

Then, by the well-known formula of linear regression, we obtain

$$p = -\,1.419{,}386, \quad q = -\,0.425{,}828,$$

the multiple correlation coefficient, between $\Delta^4\rho_u$ on the one hand, and $\Delta^2\rho_{u+1}$ and ρ_{u+2} on the other, being $r_{1.23}=0.986$. The correlation is thus very high and so it is quite reasonable to omit ϵ in (6) and to treat the resulting approximate equation according to the rules of the calculus of finite differences. We find thus that the solution of this equation is the sum of two sinusoids with the periods

$$L_1 = 9.40, \quad L_2 = 6.04.$$

Let us, then, divide our series $(\rho'_{11}, \rho'_{12}, \cdots \rho'_{64})$ into three parts, of 18 items each, and let us find, for each part separately, two sinusoids with the periods $L_1=9, L_2=6$, these being the whole numbers nearest to the theoretical values just obtained. We find the results given in Table 5.

TABLE 5

	Part I		Part II		Part III	
	$L_1=9$	$L_2=6$	$L_1=9$	$L_2=6$	$L_1=9$	$L_2=6$
Amplitude	0.19790	0.07820	0.12614	0.12527	0.29123	0.13340
Phase	231°37′	52°39′	218°2′	295°2′	270°52′	2°33′

A glance at Fig. 15 shows that the theoretical curves fit the empirical points very satisfactorily and it seems fairly certain that, if we had included one or two sinusoids more, we could have obtained a quite satisfactory fit, even if treating our empirical series as a whole. This can be proved by the fact that the sum of three sinusoids

$$\begin{aligned} z = {} & 0.1893 \sin\,[(360°/8.80)t - 47°3'] \\ & + 0.1000 \sin\,[(360°/7.14)t + 168°24'] \\ & + 0.0794 \sin\,[(360°/5.87)t - 76°3'], \end{aligned}$$

though found by rather a rough graphical estimate, fits our empirical curve in a fairly satisfactory manner.[27]

VII. THE LAW OF THE SINUSOIDAL LIMIT

Many tendencies dealt with rather empirically in the preceding discussion will be more clearly understood, and their significance more fully appreciated, if we take into consideration the following propositions.[28]

THEOREM A: (*The Law of the Sinusoidal Limit*)

Let $y_1, y_2, \cdots$ *be a chance series fulfilling the conditions,*

$$E(y_i) = 0, \quad E(y_i^2) = \sigma^2 = f(n),$$

$$\frac{E(y_i y_{i+t})}{E(y_i^2)} = r_t = \phi(t, n),$$

where n is a parameter specifying the series as a whole, and $f(n)$ and $\phi(t, n)$ are independent of i. If, furthermore, the correlation coefficient, r_1, between y_i and y_{i+1}, satisfies the condition

$$|r_1| \leqq c < 1, \qquad (n \to \infty),$$

and the correlation coefficient between $\Delta^2 y_i$ and y_{i+1}, that is, ρ_1, is such that

$$\lim \rho_1 = -1, \qquad (n \to \infty),$$

then (1) ϵ and η being taken arbitrarily small and s arbitrarily large, there will exist a number, n_0, such that for every $n > n_0$, the probability, that the absolute values of the deviations of $y_i, y_{i+1}, \cdots y_{i+s}$ from a certain sinusoid will not exceed $\epsilon\sigma$, will be $>1-\eta$; (2) the period of this sinusoid will be determined by the equation

$$\cos (2\pi/L) = r_1;$$

(3) the number of the periods in the interval $(i, i+s)$ will be arbitrarily large provided s and n be taken large enough.

This proposition (for its proof see Appendix, Section 4) would be of no interest could we not give at least a single instance of a chance series satisfying the conditions of Theorem A. This is done by

[27] The above illustration seems to throw some light on the difficulties connected with the idea of a correlation periodogram. Cf. Dinsmore Alter, "A Group or Correlation Periodogram," etc., *Monthly Weather Review*, Vol. 55, No. 6, June, 1927, pp. 263–266; Sir Gilbert Walker, "On Periodicity in Series of Related Terms," *Proc. Royal Soc.*, Ser. A, Vol. 131, No. A 818, 1931, pp. 518–532.

[28] E. Slutsky, "Sur un théorème limite relatif aux séries des quantités éventuelles," *Comptes Rendus*, Paris, t. 185, seance du 4 Juli, 1927, p. 169.

THEOREM B: Let $x_1, x_2, \cdots$ be a random series fulfilling the conditions

$$E(x_i) = 0,\ E(x_i^2) = \sigma_x^2 = \text{const.},\ E(x_i x_j) = 0, \qquad (i \neq j).$$

Now, if we put

$$x_i^{(1)} = x_i + x_{i-1},\ x_i^{(2)} = x_i^{(1)} + x_{i-1}^{(1)},\ \cdots,$$

$$x_i^{(n)} = x_i^{(n-1)} + x_{i-1}^{(n-1)},$$

and

$$y_i = \Delta^m x_i^{(n)},$$

then the series $y_1, y_2, \cdots$ will tend to obey the law of the sinusoidal limit, provided m and n be increasing indefinitely and m/n = constant (for the proof see Appendix, Section 4).

Both propositions can be generalized to the case of a chance series practically coinciding, not with one sinusoid, but with the sum of a certain number of sinusoids.[29] In every case, however, the practical coincidence (and it is a very essential character of the series under consideration) does not extend itself to the series as a whole, the respective sinusoids of closest fit being different for different partial series. This is plainly evident for the chance series of Theorem *B*, for, s being arbitrarily large and n and m being sufficiently large, the values y_i and y_{i+t} will be wholly independent of each other as soon as $t>m+n+1$, whence it follows that the phases and the amplitudes of the sinusoids practically coincident with the partial series, $y_i, y_{i+1}, \cdots y_{i+s}$, and $y_{i+t+s}, y_{i+t+s+1}, \cdots y_{i+t+2s}$, respectively, will also be independent of each other provided $t>m+n+1$.

The following considerations will show us the same problem from a somewhat different standpoint. Let us suppose a certain mechanism is being subjected to damped vibrations of a periodic character and to casual disturbances accumulating energy just sufficient to counterbalance the damping.[30] Then the movement of the system could be regarded as consisting of the two parts: of the vibrations determined by the initial conditions at some given moment, and of the vibrations generated by the disturbances that have occurred since. As soon as the first part has been nearly extinguished by the damping process after due lapse of time, the actual vibrations will be reduced practically to the second part, that is, to the accumulated consequences of the chance

[29] V. Romanovsky, "Sur la loi sinusoidale limite," *Rend. d. Circ. mat. di Palermo*, Vol. 56, Fasc. 1, 1932, pp. 82–111; V. Romanovsky, "Sur une generalisation de la loi sinusoidal limite," *Rend d. Circ. mat. di Palermo*, Vol. 57, Fasc. 1, pp. 130–136; cf. Sir Gilbert Walker, *op. cit.*, in note 27.

[30] Cf. G. Udney Yule, "On a Method of Investigating Periodicities in Disturbed Series with Special Reference to Wolfer's Sunspot Numbers," *Phil. Trans. Roy. Soc. of London*, Ser. A. Vol. 226, 1927, pp. 267–298.

causes. The latter, after a due time, being again reduced to a value not different practically from zero, the vibrations will consist of the disturbances accumulated during the second interval of time and so on. It is evident that the vibrations ultimately will have the character of a chance function, the described process being a particular instance of the summation of random causes. Should the disturbances be small enough, there would exist an arbitrarily large, but finite, number, L_0, such that the resulting process would be practically coincident, in every interval of the length, $L \leqq L_0$, with a certain periodic (or nearly periodic) function, obeying thus the law of the sinusoidal limit.

Analogous considerations may be applied to the motion of planetary, or star systems, the innumerable cosmic influences being considered as casual disturbances. The paths of the planets, if regarded during billions of years, should be considered, therefore, as chance functions, but if we do not wish to go beyond thousands of years their approximate representation must be taken as not casual.

The chance functions of the type just considered appearing on the one end of the scale, and the random functions on the other, there evidently must exist all possible intermediate gradations between these extremes. The ability of the coherent chance series to simulate the periodic, or the nearly periodic, functions, seems thus to be definitely demonstrated.

It remains for us to try to clear up theoretically the remarkable property of some specimens of chance series, which do not belong to the extreme classes of their type, of being approximately representable by a small number of sinusoids, over a shorter or longer interval.

It is well known that every empirical series consisting of a finite number of terms ($N=2n$ or $2n+1$) can be represented precisely by a finite Fourier series, that is by the sum of a finite number (n) of sinusoids. Further, it is plainly evident, the series under consideration being chance series and the coefficients of the Fourier expansion,

$$y_t = A_0 + \sum_1^n A_k \cos (2\pi kt/N) + \sum_1^n B_k \sin (2\pi kt/N),$$

that is, the values $A_0, A_1, \cdots A_n$, and $B_0, B_1, \cdots B_n$, being linear functions of $y_1, y_2, \cdots y_n$, that the variables A_k and B_k will also be chance variables. Their mathematical expectations, standard deviations, and the correlation coefficients between them can be easily obtained.[31] Denoting by R_k^2 the intensity of the kth harmonic, that is,

[31] E. Slutsky, "Alcuni applicazioni di coefficienti di Fourier al analizo di sequenze eventuali coherenti stazionarii," *Giorn. d. Istituto Italiano degli Attuari*, Vol. 5, No. 4, 1934; see also E. Slutsky, "Sur l'extension de la theorie de periodogrammes aux suites des quantités dependentes." *Compte Rendus*, t. 189, seance du 4 novembre, 1929, p. 722.

the square of its amplitude, we shall have

$$R_k^2 = A_k^2 + B_k^2,$$

and

$$(8)\quad E(R_k^2) = (4\sigma_y^2/N)\left[1 + 2\sum_1^{N-1} r_t \cos(2\pi kt/N)\right] - (8\sigma_y^2/N^2)\sum_1^{N-1} tr_t \cos(2\pi kt/N),$$

whence, for the case of a random series, we obtain at once the formula of Schuster,

$$(9)\quad E(R_k^2) = 4\sigma_y^2/N,$$

the probability distribution being the same in both cases,

$$(10)\quad P(R_k^2 > Z^2) = \exp\left[-Z^2/E(R_k^2)\right].$$

Let us suppose the m intensities happening to have the largest values in some given case are those with the indices: $\alpha, \beta, \cdots \mu$ and let

$$\tfrac{1}{2}(R_\alpha^2 + R_\beta^2 + \cdots + R_\mu^2) = ps^2,$$

s^2 being the square of the empirical standard deviation and p the coefficient measuring the degree of approximation reached in the given case by means of m harmonics. By taking account of (8), (9), and (10), we see at once that, in the case of a random series, the indices $\alpha, \beta, \cdots \mu$ are able to assume any values with equal probability but that in the case of a coherent series those having the largest values of $E(R_k^2)$ will be the most probable. As half of the sum of the intensities is equal to the square of the empirical standard deviation (Parceval's theorem), it is but natural that the coherent chance series, in many cases at least, may be represented—the degree of approximation being the same—by a smaller number of harmonics than the random series.

It can be proved further (under suppositions of a not very restrictive character) that the correlation coefficients between the intensities belonging to the same interval, as well as between those belonging to the adjacent intervals, are quantities of the order $1/N^2$, and that the standard deviation of the intensity, σ_R^2, tends to be equal to its probable value, $E(R_k^2)$. Whence it is evident that the indices of the harmonics which happen to be the most suited for the representation of the series in a certain interval must also be practically independent of the indices of the "best" harmonics in adjacent intervals, the length of these intervals being sufficiently large. The larger the probable value of the intensity the larger also must be the extent of its casual variation. These are properties quite consistent with a considerable degree of regularity—as well as with the abrupt changes of the "regimes" determined by studying the empirical series dealt with in the foregoing pages.

APPENDIX

MATHEMATICAL NOTES OF THE THEORY OF RANDOM WAVES

1. Let $x_0, x_1, \cdots x_i, \cdots$ be a random series, that is, a series of chance values independent of each other. Let this be our basic series and let it be considered as a model of incoherent series of random causes. Denoting by the symbol E the mathematical expectation, let us suppose that

$$E(x_i) = 0,\ E(x_i^2) = \sigma_x^2 = \text{const.};\ E(x_i x_j) = 0, \qquad (i \neq j). \tag{1}$$

From the basic incoherent series of causes let us construct a coherent series of "consequences," $\cdots y_{i-2}, y_{i-1}, y_i, \cdots$, by the scheme

$$y_i = \sum_{k=0}^{n-1} A_k\, x_{i-k}, \tag{2}$$

where the quantities A_k are constants.[32] Then, by using (1) and (2), it can easily be shown that

$$E(y_i) = 0, \tag{3}$$

$$E(y_i^2) = \sigma_y^2 = \sigma_x^2 \sum_{k=0}^{n-1} A_k^2, \tag{4}$$

$$E(y_i y_{i+t}) = \sigma_x^2 \sum_{k=0}^{(n-1)-t} A_k A_{k+t}. \tag{5}$$

Since equations (4) and (5) do not depend on i, the coefficient of correlation between y_i and y_{i-t}, that is, r_t, is also independent of i, and we have

$$r_t = \frac{\sum_{k=0}^{(n-1)-t} A_k A_{k+t}}{\sum_{k=0}^{n-1} A_k^2}, \tag{6}$$

from which it immediately follows that

$$r_0 = 1;\ r_t = r_{-t};\ r_t = 0,\ (t \geqq n). \tag{7}$$

The process of moving summation can be repeated. As before, let us take $\cdots x_{i-2}, x_{i-1}, x_i \cdots$ as the basic series underlying the conditions

[32] Cf. Prof. Birger Meidell's valuable investigation of the analogous cumulative processes in his paper, "Über periodische und angenäherte Beharrungszustände," *Skandinavisk Aktuarietidskrift*, 1926, p. 172.

(1). Then on performing an s-fold moving summation we obtain the following successive consequence series:

$$
(8)\qquad
\begin{aligned}
x_i^{(1)} &= \sum_{k=0}^{n-1} \alpha_k^{(1)} x_{i-k}; \quad x_i^{(2)} = \sum_{k=0}^{n-1} \alpha_k^{(2)} x_{i-k}^{(1)}; \cdots \\
x_i^{(s)} &= \sum_{k=0}^{n-1} \alpha_k^{(s)} x_{i-k}^{(s-1)}.
\end{aligned}
$$

After the s-fold summation we have an expression of type (2) with $y_i = x_i^{(s)}$. Hence, if we take $n=2$ and $\alpha_k^{(j)}=1$, it can easily be shown that

$$
(9)\qquad y_i = x_i^{(s)} = \sum_{k=0}^{s} C_k^s x_{i-k}.
$$

If we put

$$
(10)\qquad \phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}t^2\right),
$$

we can, by the use of well-known transformations, obtain the approximate expression (which we write for an *even* s)

$$
(11)\qquad y_i = D \sum_{k=0}^{s} x_{i-k} \phi\left(\frac{k - s/2}{\sqrt{s/4}}\right),
$$

D being a coefficient, the value of which need not concern us here.

It is very remarkable that a similar result will always be obtained for s sufficiently great, whatever be the weights used, provided it is supposed (1) that the weights are not negative, (2) that they remain constant at every given stage of the process, and (3) that the summation does not tend to degenerate into a mere repetition of the same values, which would be the case should all $\alpha_k^{(s)}$ but one tend to approach 0; (the sum of the weights is supposed, without loss of generality, to be constant).

To prove this, let us remark first that the result of the s-fold summation given by (8) can evidently be obtained by a similar s-fold summation with the weights

$$
p_0^{(j)}, p_1^{(j)}, \cdots, p_{n-1}^{(j)}, \qquad (j = 1, 2, \cdots, s),
$$

where

$$
p_k^{(j)} = \frac{\alpha_k^{(j)}}{m_j}, \quad m_j = \sum_{k=0}^{n-1} \alpha_k^{(j)},
$$

if we multiply the resulting weights by the proportionality factor $m_1 \cdot m_2 \cdot \cdots \cdot m_s$.

Now to prove our proposition we shall use the following analogy (kindly suggested to the author by Prof. A. Khinchin).

Let $z_1, z_2, \cdots z_s$ be a set of random variables whose possible values are 0, 1, 2, $\cdots$, $n-1$, the respective probabilities being $p_0^{(j)}$, $p_1^{(j)}$, $\cdots p_{n-1}^{(j)}$, $(j=1, 2, \cdots s)$. Then it is easy to see that the probability of the equation

$$k = z_1 + z_2 + \cdots + z_s \tag{12}$$

must be equal to the coefficient of x^k in the expansion of

$$\prod_{j=1}^{s} (p_0^{(j)} + p_1^{(j)}x + p_2^{(j)}x^2 + \cdots + p_{n-1}^{(j)}x^{n-1}). \tag{13}$$

On the other hand, it can be proved that the same coefficient, multiplied by $m_1 \cdot m_2 \cdot \cdots \cdot m_s$, will be equal to the coefficient A_k in the equation (2) obtained by an s-fold summation according to the scheme (8) with the weights

$$m_j p_0^{(j)},\ m_j p_1^{(j)},\ \cdots,\ m_j p_{n-1}^{(j)}, \qquad (j = 1, 2, \cdots, s).$$

This is easily seen for $s=2$ and the result can be generalized by mathematical induction from s to $s+1$.

This analogy leads us to the following considerations. Let us put

$$\begin{cases} a_j = E(z_j) = \sum_{k=0}^{n-1} k p_k^{(j)}, \\ b_j = E[(z_j - a_j)^2] = \sum_{k=0}^{n-1} (k - a_j)^2 p_k^{(j)}, \\ c_j = E[|z_j - a_j|^g] = \sum_{k=0}^{n-1} |k - a_j|^g p_k^{(j)}, \qquad (g > 2). \end{cases} \tag{14}$$

It is evident that $b_j=0$ only if every $(k-a_j)^2 p_k^{(j)}=0$ for $k=0, 1, 2, \cdots (n-1)$, and that this is possible only when every $p_k^{(j)}$ but one is equal to zero, the exceptional p being 1, in which case the values $x_k^{(j)}$ are merely repetitions of the values $x_k^{(j-1)}$.

This case being excluded, we shall have, on the average at least,

$$(1/s)\sum_{1}^{s} b_i > \epsilon > 0; \tag{15}$$

whence

$$\frac{\left[\sum_{1}^{s} c_i\right]^2}{\left[\sum_{1}^{s} b_i\right]^g} < \frac{\left[(1/s)\sum_{1}^{s} c_i\right]^2}{s^{g-2}\epsilon^g} \to 0. \tag{16}$$

But this is the well known Liapounoff's condition, under which the probability distribution of the sum $z_1+z_2+\cdots+z_s$, that is, the distribution of the coefficients A_k, tends to the normal law.[33]

Let us put, for instance,

$$\alpha_0^{(j)} = \alpha_1^{(j)} = \cdots = \alpha_{n-1}^{(j)} = 1, \qquad (j = 1, 2, \cdots, s).$$

Then we obtain

$$\begin{aligned} m_j &= \sum_{k=0}^{n-1} \alpha_k^{(j)} = n, \\ m_1 m_2 \cdots m_s &= n^s, \\ p_k^{(j)} &= 1/n, \end{aligned} \tag{17}$$

for

$$j = 1, 2, \cdots, s, \quad \text{and} \quad k = 0, 1, \cdots, n-1;$$

and

$$\begin{aligned} a_j &= E(z_j) = \sum_{k=0}^{n-1} k p_k^{(j)} = (n-1)/2, \\ b_j &= E[(z_j - a_j)^2] = (1/n)\left\{ \sum_{k=0}^{n-1} k^2 - n a_j^2 \right\} = (n^2-1)/12. \end{aligned} \tag{18}$$

Whence

$$k_0 = E(k) = E\left[\sum_1^s z_j \right] = s(n-1)/2, \tag{19}$$

and

$$\sigma_k = \sqrt{s b_j} = \sqrt{s(n^2-1)/12}. \tag{20}$$

As s tends towards ∞, the value of A_k will thus approach a limit, which enables us to write, for s large but finite, the following approximate equations:[34]

[33] It is evident that, since Liapounoff's theorem is a proposition about the limit properties of certain integrals and not of the individual ordinates, the above demonstration must be interpreted also in the same sense. For many cases, however, for example, in the case of the illustration below, the additional conditions are satisfied under which the values of the variables A_k themselves are tending toward the ordinates of the Gaussian curve.

Cf. Liapounoff, "Nouvelle forme du théorème sur la limite de probabilité," *Memoires de l'Academie de science de St.-Petersbourg*, serie 8, Vol. 12, No. 5.

R. von Mises, *Vorlesungen aus dem Gebiete der Angewandten Mathematik*, Bd. I—*Wahrscheinlichkeitsrechnung und ihre Anwendungen*, 1931, p. 200–212.

R. von Mises, "Generalizzazione di un teorema sulla probabilità della somma di un numero illimitato di variabili casuali," *Giornale dell'Istituto Italiano degli Attuari*, Anno 5, N4, p. 483–495.

[34] This result coincides with that given in the first edition of this memoir in 1927; it was supplied to the author by the courtesy of Prof. A. Khinchin who derived it by the application of the well-known Cauchy theorem to the evaluation of the coefficient of x^k in the expansion of

$$(1 + x + x^2 + \cdots + x^{n-1})^s.$$

I am sorry that the calculations are too long to be reproduced here.

(21) $A_k \overset{a}{=} n^s\sqrt{6/\pi s(n^2-1)}\exp\{-6(k-k_0)^2/s(n^2-1)\}.$

For the general case, we shall give here the following illustration. Let the weights for a set of successive summations be certain random numbers. For this purpose, let us choose consecutive groups of three numbers from the first basic series (Column 2, Table I, Appendix II). For the first moving summation the weights will be $\alpha_0^{(1)}=5$, $\alpha_1^{(1)}=4$, $\alpha_2^{(1)}=7$; for the second $\alpha_0^{(2)}=3$, $\alpha_1^{(2)}=0$, $\alpha_2^{(2)}=3$, etc. Performing the substitutions indicated by formula (8) we obtain the resulting weights corresponding to A_k of formula (2), $A_k^{(1)}=\alpha_k^{(1)}$, $A_k^{(2)}$, $A_k^{(3)}$, $\cdots$ $A_k^{(10)}$. For each given s, we divide the weights by the largest $A_k^{(s)}$ to obtain the relative weights, $A'_k{}^{(s)}$ (see Table VIII, Appendix II, of the original paper, and Figure 5). The series of quantities $A'_k{}^{(10)}$ does not differ greatly from the Gaussian curve obtained by putting[35]

(22) $$B'_k{}^{(10)} = 1004\exp\{-\tfrac{1}{2}[(k-9.26)/2.67]^2\}.$$

2. The coefficients of correlation between the terms of a coherent series are, in many cases, easy to obtain by using formula (6). For a simple moving summation of n equally weighted items at a time, we have $A_0=A_1=\cdots=A_{n-1}=1$. It is easy to see that

(23) $$\begin{cases} r_t = (n-|t|)/n, & (|t| \leqq n) \\ r_t = 0, & (|t| \geqq n). \end{cases}$$

From formulas (4), (5), (9) and the properties of $\overset{s}{C}_k$, we find for the s-fold moving summation of two terms, that is, $(n=2)$, that

(24) $$\sigma_y^2 = \sigma_x^2[1+(\overset{s}{C}_1)^2+(\overset{s}{C}_2)^2+\cdots+(\overset{s}{C}_{s-1})^2+1] = \sigma_x^2\overset{2s}{C}_s,$$

and

(25) $$E(y_iy_{i+t}) = \sigma_x^2[\overset{s}{C}_0\overset{s}{C}_t+\overset{s}{C}_1\overset{s}{C}_{t+1}+\cdots+\overset{s}{C}_{s-t}\overset{s}{C}_s] = \sigma_x^2\overset{2s}{C}_{s-t}.$$

Hence

(26) $$r_t = \overset{2s}{C}_{s-t}/\overset{2s}{C}_s = \frac{s(s-1)\cdots(s-t+1)}{(s+1)(s+2)\cdots(s+t)}.$$

Consider another case. Let us form, from a basic series, a coherent series by the scheme:

[35] Let us pass a second degree parabola through $A_0'^{(10)}$, $A_1'^{(10)}$, $A_2'^{(10)}$; another through $A_2'^{(10)}$, $A_3'^{(10)}$, $A_4'^{(10)}$; etc. Denote the area of this figure by S, its maximum ordinate by y_0, and the abscissa bisecting the area by k_0. Then, in the Gaussian equation

$$B'_k{}^{(10)} = [S/\sigma\sqrt{2\pi}]\exp\{-\tfrac{1}{2}[(k-k_0)/\sigma]^2\},$$

all of the parameters are known, since

$$k_0 = 9.26,\ y_0 = S/\sigma\sqrt{2\pi} = 1004,$$

and hence

$$\sigma = S/y_0\sqrt{2\pi} = 2.67$$

$$y_i = D \sum_{k=0}^{2k_0} x_{i-k} \phi[(k - k_0)/\sigma], \tag{27}$$

where $\phi(t)$ is given by formula (10), D is a constant, and k_0 is a number large enough so that $\phi(t)$ can be neglected for $|t| > k_0/\sigma$. Then, from (6), the coefficient of correlation is

$$r_t = \frac{\sum_{k=0}^{2k_0 - t} \phi[(k - k_0)/\sigma] \phi[(k - k_0 + t)/\sigma]}{\sum_{k=0}^{2k_0} \{\phi[(k - k_0)/\sigma]\}^2}. \tag{28}$$

If σ is sufficiently large we can substitute integrals for the summations in (28). Inserting $z = k - k_0$, we then obtain the approximation formula

$$r_t = \frac{\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2}\frac{z^2 + (z + t)^2}{\sigma^2}\right] dz}{\int_{-\infty}^{+\infty} \exp\left[-\frac{z^2}{\sigma^2}\right] dz} \tag{29}$$

$$= \exp(-t^2/4\sigma^2) = \frac{\phi(t/\sigma\sqrt{2})}{\phi(0)}.$$

Inasmuch as Model III is formed by the scheme of formula (27), with $D = 10^4$, $k_0 = 48$, and $\sigma = 10$, we can calculate the correlation function by formula (29). The values for $[\phi(t/\sqrt{200})]/\phi(0)$, $(t = 0, 1, 2, \cdots)$, were calculated with the aid of Sheppard's tables.[36] The symbol $R_{t(III)}$, instead of $r_{t(III)}$, indicates that an approximate, and not an exact, formula was used in the calculation.

Model IVa was obtained by a 12-fold moving summation of two items; therefore, its correlation function, $r_{t(IVa)}$, is obtained by using formula (26), which, if we consider (11), gives $R_{t(IVa)} = \phi(t/\sqrt{6})/\phi(0)$. The discrepancies between the two results are rather small. The correspondence between $r_{t(IVb)}$ and $R_{t(IVb)}$ is somewhat less, as is also that between $r_{t(IVc)}$ and $R_{t(IVc)}$. Both sets were computed by formula (45) (see next paragraph), but for the calculation of $r_{t(IVb)}$ and $r_{t(IVc)}$ the actual values of $r_{t(IVa)}$ were used as the base, while for $R_{t(IVb)}$ and $R_{t(IVc)}$ the approximate values of $R_{t(IVa)}$, obtained by the Gaussian formula, were used. Even here the discrepancies are not very great when regarded from the same point of view (see Figure 6).

[36] *Tables for Statisticians and Biometricians*, ed. by K. Pearson, Cambridge, 1914, Table II.

Finally, for Model II, corresponding to the scheme

$$(30)\qquad y_i = x_i^{(2)} = \sum_{k=0}^{18} A_k^{(2)} x_{i+k} + 50,$$

$$(A_k^{(2)} = 1, 2, \cdots, 9, 10, 9, \cdots, 2, 1),$$

the coefficients of correlation can be obtained directly from formula (6). It is worth noting that, even in this case, a good approximation, $R_{t(II)}$, can be obtained by the use of the Gaussian curve, the equation being

$$(31)\qquad R_{t(II)} = \exp\left[-\tfrac{1}{2}(t/5.954)^2\right],$$

where $\sigma = 5.954$ was obtained by equating the areas of the Gaussian curve and of the empirical curve, and the computations were carried through with the help of Simpson's rule (see Figure 6).

A few more words may be said about the correlational function for the weights, $A'^{(10)}_k$, of our example of the crossing of random weights (see Section 2 of the text, Appendix, Section 1, and Figure 6). The exact values of the coefficients of correlation (r_t) were found by formula (6), while approximate values (R_t) were obtained from the equation

$$(32)\qquad R_t = \exp\left[-\tfrac{1}{2}(t/3.727)^2\right],$$

which was obtained in the same manner as was equation (31). Both the exact and approximate values are given in Table 1 (see also Figure 6). Also, let us note that, from the equation of the Gaussian curve which approximates the weights, $A'^{(10)}_k$ (see formula (22) above), it is possible to find an approximate expression for the coefficients of correlation by using formula (29). An expression analogous to (32) would be obtained, but instead of $\sigma = 3.727$, we would have $\sigma = 3.776$. The correlation coefficients are only slightly less accurate than those found from formula (32), the deviations are all of one sign, and none is greater than 0.009.

Let us make one more observation. If a chance variable $y_i = u_i + v_i$, where u_i is a coherent series and v_i is a random series, it is easy to show that

$$(33)\qquad E(y_i y_{i+t}) = E(u_i u_{i+t}),$$

$$(34)\qquad (Ey_i^2) = \sigma_u^2 + \sigma_v^2,$$

$$(35)\qquad r_{y_i, y_{i+t}} = \frac{r_{u_i, u_{i+t}}}{1 + (\sigma_v^2/\sigma_u^2)},$$

where $E(u_i)$ and $E(v_i)$ are taken equal to zero.

If $r_{u_i, u_{i+t}}$ lies along the Gaussian curve, then $r_{y_i, y_{i+t}}$ will lie along a similar curve with ordinates proportionally reduced, except that r_0 will, as formerly, equal unity; the *chapeau de gendarme* has taken on the spike of the *budenovka* (a Soviet military cap). It is to be expected that this figure and the analogous figures for the correlation function of the differences (formula (45) of the following paragraph) will be encountered in the investigation of empirical series.[37]

3. Let us now investigate the differences of various orders of the series y_i, i.e., $\Delta^\alpha y_i$, $\Delta^\beta y_i$, and their coefficients of correlation.[38]
As before, let

$$(36) \qquad E(y_i) = 0; \; E(y_i^2) = \sigma_y^2 = \text{constant}; \; E(y_i y_{i+t})/\sigma_y^2 = r_t,$$

where r_t is supposed to be independent of i. Let us introduce the notation

$$(37) \qquad r_t^{(\alpha,\beta)} = r_{\Delta^\alpha y_i, \Delta^\beta y_{i+t}}, \qquad (\alpha \geqq \beta);$$

and, in particular,

$$(38) \qquad r_t^{(\alpha,\alpha)} = r_{\Delta^\alpha y_i \Delta^\alpha y_{i+t}} \qquad r_t^{(\alpha,0)} = r_{\Delta^\alpha y_i, y_{i+t}}.$$

By using the equality

$$(39) \qquad C_k^{2\alpha} = C_{\alpha-k}^{\alpha} C_0^{\alpha} + C_{\alpha-(k-1)}^{\alpha} C_1^{\alpha} + \cdots + C_{\alpha-1}^{\alpha} C_{k-1}^{\alpha} + C_\alpha^{\alpha} C_k^{\alpha},$$

it can be shown that

$$\sigma_{\Delta^\alpha y_i}^2 = E[(\Delta^\alpha y_i)^2] = E[(C_\alpha^\alpha y_{i+\alpha} - C_{\alpha-1}^\alpha y_{i+\alpha-1} + C_{\alpha-2}^\alpha y_{i+\alpha-2}$$
$$(40) \qquad - \cdots + (-1)^\alpha C_0^\alpha y_i)^2] = (-1)^\alpha \Delta^{2\alpha} r_{-\alpha} \sigma_y^2.$$

From (40), by using the equality,

$$(41) \qquad C_k^{\alpha+\beta} = C_{\alpha-k}^{\alpha} C_0^{\beta} + C_{\alpha-k+1}^{\alpha} C_1^{\beta} + \cdots + C_{\alpha-1}^{\alpha} C_{k-1}^{\beta} + C_k^{\beta},$$

we obtain

$$(42) \qquad E[\Delta^\alpha y_i \Delta^\beta y_{i+t}] = (-)^\alpha \Delta^{\alpha+\beta} r_{t-\alpha} \sigma_y^2, \qquad (\alpha \geqq \beta),$$

and from this we have

$$(43) \qquad r_t^{(\alpha,\beta)} = \frac{(-1)^\alpha \Delta^{\alpha+\beta} r_{t-\alpha}}{\sqrt{(-1)^{\alpha+\beta} \Delta^{2\alpha} r_{-\alpha} \Delta^{2\beta} r_{-\beta}}}.$$

In the same manner we obtain

$$(44) \qquad r_t^{(\alpha,0)} = \frac{(-1)^\alpha \Delta^\alpha r_{t-\alpha}}{\sqrt{(-1)^\alpha \Delta^{2\alpha} r_{-\alpha}}},$$

[37] Cf. Figure 19 of Yule, "Why Do We Sometimes Get Nonsense-Correlations · · · ," *loc. cit.*, p. 43.

[38] Cf. O. Anderson, "Über ein neues Verfahren bei Anwendung der 'Variate-Difference' Methode," *Biometrika*, Vol. 15, 1923, pp. 142 ff.

and, as a special case of formula (43), we have

$$r_t^{(\alpha,\alpha)} = \frac{\Delta^{2\alpha} r_{t-\alpha}}{\Delta^{2\alpha} r_{-\alpha}}. \tag{45}$$

By this formula and by (26) the correlation coefficients for Models IV*b* and IV*c* have been computed.

4. Let us now prove Theorem A (see Section VII above). The regression coefficient of $\Delta^2 y_i$ on y_{i+1} being

$$E(\Delta^2 y_i \cdot y_{i+1})/\sigma_y^2 = -2(1 - r_1),$$

we obtain the approximate equation

$$\Delta^2 y_i \overset{a}{=} -2(1 - r_1) y_{i+1}.$$

Whence

$$y_{i+2} \overset{a}{=} 2r_1 y_{i+1} - y_i, \tag{46}$$

the errors of both equations being evidently identical. If we denote this error by α_2 and put $\beta_2 = \alpha_2/\sigma_y$, we may apply the well-known formula of correlation theory and thus obtain

$$E(\beta_2^2) = E(\Delta^2 y_i)(1 - \rho_1^2)/\sigma_y^2 = (1 - \rho_1^2)(6 - 8r_1 + 2r_2). \tag{47}$$

Now, under the suppositions of Theorem A,

$$\lim_{n \to \infty} \beta_2^2 = 0;$$

whence, applying Tchebycheff's theorem, we see that β_2 has the stochastical limit $E(\beta_2) = 0$, $(n \to \infty)$; that is, ϵ and η being arbitrarily small, the probability

$$P\{ |\beta_2| > \epsilon \} < \eta,$$

provided n is sufficiently large.

On the other hand, if we put

$$\begin{aligned} y_{i+2} &= 2r_1 y_{i+1} - y_i + \alpha_2, \\ y_{i+3} &= 2r_1 y_{i+2} - y_{i+1} + \alpha_3, \\ &\cdots\cdots\cdots\cdots \\ y_{i+s} &= 2r_1 y_{i+s-1} - y_{i+s-2} + \alpha_s; \end{aligned}$$

and if we insert y_{i+2} in the second equation of this system, y_{i+3} in the third, and so on, we obtain, after reduction,

$$y_{i+s} = C_1 y_{i+1} + C_2 y_i + \lambda_s \sigma_y, \tag{48}$$

where

$$\lambda_s = a_0\beta_s + a_1\beta_{s-1} + \cdots + a_{s-2}\beta_2, \tag{49}$$

the values $a_0, a_1, \cdots a_{s+2}$, being determined by the conditions

$$\begin{cases} a_0 = 1, \quad a_1 = 2r_1, \\ a_{k+2} = 2r_1a_{k+1} - a_k. \end{cases} \tag{50}$$

This equation is identical with

$$y_{i+2} = 2r_1y_{i+1} - y_i, \tag{51}$$

that is, with (46) considered as a precise equation. The solution of (51) or (50) can be obtained easily. We find

$$y_t = A \cos (2\pi t/L) + B \sin (2\pi t/L), \tag{52}$$

and

$$a_k = C \cos (2\pi k/L) + D \sin (2\pi k/L), \tag{53}$$

where we have put

$$\cos (2\pi/L) = r_1, \tag{54}$$

L being the period of the respective sinusoid. It is evident that, under the assumptions of Theorem A, ($|r_1| \leqq \lambda < 1$), we shall have

$$L \leqq 2\pi/\text{arc} \cos \lambda = H = \text{constant}. \tag{55}$$

Two sinusoids must now be considered. The first, which will be denoted by S_1, is determined by (51), or (52), and the initial points y_i, y_{i+1}. It is evident that $C_1y_{i+1}+C_2y_i$ in (48) is the ordinate of S_1 which could be obtained in this form from (51) by successive substitutions. The deviation of the actual value of y_{i+s} from S_1 is $\lambda_s\sigma_y$ as given by (48) and (49), the coefficients a_k being the ordinates of the second sinusoid (S_2) determined by (50) or (53), and the initial values $a_0=1$, $a_1=2r_1$. But, if we put, in (53), $k=0$, and then $k=1$, we obtain

$$C = 1, \quad D = r_1/\sqrt{1 - r_1^2}.$$

Hence the amplitude of S_2 is

$$\sqrt{C^2 + D^2} = 1/\sqrt{1 - r_1^2} \leqq 1/\sqrt{1 - \lambda^2} = K = \text{constant}.$$

$$C = 1, \quad D = r_1/\sqrt{1 - r_1^2}.$$

Thus, taking into account that every a_k in (49) has an upper limit $\leq\sqrt{C^2+D^2}$, $(n\to\infty)$ and remembering the theorems of my *Metron* memoir[39] we conclude that λ_s has the stochastical limit $=0$ and that, ϵ and η being arbitrarily small and s arbitrarily large, the probability that the conditions

$$\lambda_2 < \epsilon,\ \lambda_3 < \epsilon,\ \cdots,\ \lambda_s < \epsilon,$$

are simultaneously satisfied will be $>1-\eta$ provided n is sufficiently large. The formulas (54) and (55) complete the proof.

To prove theorem B, we may proceed here as follows: It is seen by (43) and (45) that the correlation coefficient between y_i and y_{i+1} is

$$r_1 = r_1^{(m,m)} = \frac{\Delta^{2m}r_{-m+1}}{\Delta^{2m}r_{-m}}, \tag{56}$$

and the correlation coefficient between $\Delta^2 y_i$ and y_{i+1} is given by

$$\rho_1 = r_1^{(m+2,m)} = \frac{(-1)^m\Delta^{2m+m}r_{-(m+1)}}{\sqrt{\Delta^{2(m+2)}r_{-(m+2)}\Delta^{2m}r_{-m}}}, \tag{57}$$

where, using (26) we must put

$$r_t = \frac{C^{2n}_{n-t}}{C^{2n}_{n}}.$$

On the other hand, r_{-t} being equal to r_t, it can easily be seen that

$$C_n^{2n}\Delta^{2m}r_{-m} = \sum_{k=0}^{2m}(-1)^k C_k^{2m}C_{n-m+k}^{2n} = A_{n+m}, \tag{58}$$

where A_{n+m} is the coefficient of x^{n+m} in the expansion of $(1+x)^{2n}$ $(1-x)^{2m}$. Applying Cauchy's theorem we have

$$A_{n+m}\frac{1}{2\pi i}\int_{|1|}\frac{(1+x)^{2n}(1-x)^{2m}}{x^{n+m+1}}\,dx.$$

If we put $x=e^{i\phi}$, we obtain, after reduction,

$$A_{n+m} = [(-1)^m 2^{2(n+m)}/\pi]\int_0^{\pi}\cos^{2n}\phi\,\sin^{2m}\phi\,d\phi.$$

[39] E. Slutzky, "Über Stochastische Asymptoten und Grenzwerte," *Metron*, Vol. 5, N. 3, 1925, pp. 61–64.

Hence

$$(59)\quad A_{n+m} = \frac{(-1)^m 2^{n+m} 1\cdot 3 \cdots (2n-1)\cdot 1\cdot 3 \cdots (2m-1)}{1\cdot 2\cdot 3 \cdots (n+m)}.$$

Thus we obtain, by (58),

$$(60)\qquad \Delta^{2m} r_{-m} = \frac{(-1)^m 2^m 1\cdot 3 \cdots (2m-1)}{(n+1)(n+2)\cdots(n+m)}.$$

If we notice that

$$\Delta^{2m+2} r_{-(m+1)} = \Delta^{2m} r_{-(m+1)} - 2\Delta^{2m} r_{-m} + \Delta^{2m} r_{-(m-1)},$$

where, evidently,

$$\Delta^{2m} r_{-(m+1)} = \Delta^{2m} r_{-(m-1)},$$

we get

$$\Delta^{2m} r_{-m+1} = \tfrac{1}{2}\Delta^{2(m+1)} r_{-(m+1)} + \Delta^{2m} r_{-m}$$

$$(61)\qquad = \frac{(-1)^m 2^m 1\cdot 3 \cdots (2m-1)(n-m)}{(n+1)(n+2)\cdots(n+m+1)},$$

so that, by (56), (57), (60), and (61), we obtain

$$(62)\qquad r_1 = (n-m)/(n+m+1)$$

and[40]

$$(63)\qquad \rho_1 = -\sqrt{\frac{(2m+1)(n+m+2)}{(2m+3)(n+m+1)}}.$$

Now, it is evident that, n/m being constant,

$$r_1 < \frac{n/m - 1}{n/m + 1} < 1$$

and

$$\rho_1 \to -1, \qquad (n \to \infty),$$

which proves Theorem B.

CORRECTIONS OF BASIC DATA

The tables of figures which contain the series used in the present investigation are to be found in the original paper (*loc. cit.*, pp. 57–64). As the preparation of them has involved a great deal of time and labor

[40] To apply this formula in the case of No. V, we should notice that Model III is approximately equivalent to the series (y_1) of Theorem B, with $n = 400$.

and as it may be expected that someone will make use of them for the purpose of analogous studies, we give here correct readings for the *errata* found after the figures had been published. (Those relating to Table VI are immaterial and are omitted here.)

Table	Column					
	1	2	6	7	8	10
I	300				−453	
	418			−361		
	637				332	
	638			− 10	461	
	639		1367	451	295	
	807			255		
	819				211	
	820			496	−252	
	821		2290	244	−488	
	971				− 99	
	972			263	− 98	
	973		2134	165	−186	
III	23	− 4551				
	72	−20219				
IX	9		0.0007			
	11					0.0059
	13					0.0001
	14					0.0000

EUGEN SLUTZKY

Mathematical Institute of the Moscow State University
Moscow, U.S.S.R.